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On Elements Connected each to each by One or the other of Two Reciprocal Relations.

By C. De Polignac.

CHAPTER I.

Art. 1. In what follows I represent by $\overline{\overline{E}}$ an aggregate of n elements.

$$e_1 e_2 \dots\dots e_n$$

looked upon as mere ideal concepts.

Between any two elements there will exist by hypothesis a reciprocal relation R_1 or a reciprocal relation R_2 which, for pure convenience, I shall call respectively Variation and Permanence, without attaching any particular meaning to either of these terms. Except being reciprocal, these relations are undefined and unrestricted, so that if two elements are supposed to be connected to a third by the relation R_1, it does not follow that the same relation R_1 subsists between them as would be the case if, for instance, R_1 was a relation of equality or any other implying with respect to the individual element some intrinsic attribute of magnitude, form, etc.

Conventional Terms.

Any two elements e_i, e_j must, by the fundamental hypothesis, be connected either by a variation or by a permanence. If by a variation, for instance, we shall say that e_i and e_j have a variation together, or that e_j has a variation with e_i, and vice versa.

We shall also say that e_j is a variation-element of e_i, and vice versa.

It will sometimes be convenient to use the term variation instead of variation-element, thus, where no confusion of ideas can occur, we shall speak of the variations of e_i, meaning the variation-elements.

All the same expressions will apply to permanences, *mutatis mutandis.*

Cycles of Variations and Permanences.

Every pair of elements out of the aggregate $\overline{E}$, having been arbitrarily given a variation, or a permanence, if several elements $ee' \dots e''$ have nothing but variations between them, we shall say that they form *a cycle of variations.* The term *cycle of permanences* will be similarly understood.

2.—*Annotation.*

We write the elements in columns with the only restriction that any two elements must not be put down in the same column when they are connected by *one* of the two relations, say to fix ideas, the relation called variation. The law of annotation is then simply:

Any two elements between which there is a variation, must not be written in the same column.

Regard being had to this rule, a number of schemes will be formed, such as

$$\begin{array}{llll} e_1 & e_2 & e_4 & \dots e_i, \\ e_3 & e_6 & & \dots e_j, \\ e_5 & & & \dots e_h, \\ \dots & \dots & \dots & \dots \end{array}$$

each scheme containing all the elements, and, in each column, the elements giving nothing but permanences.

Observe that, in conformity with the rule and whatever be the set of relations, we shall have the scheme:

$$e_1 \quad e_2 \quad e_3 \dots e_{n-1} \quad e_n,$$

in which each column is composed of a single element.

The law of annotation is only concerned with the composition of the columns in every scheme, not with the order in which the elements are written in each column, nor with the order in which the columns are written in each scheme. It must be, therefore, well understood that two schemes are distinct only when they differ by the composition of some column or other.

3.—*Table of Schemes.*

I shall, in what follows, use the word Table to designate an aggregate of annotated schemes. The full or general Table will mean the aggregate of all the schemes possible under the law of annotation. Their number necessarily depends upon the relations arbitrarily established between the elements, that is to say, upon the incidence of variation or permanence between each and every pair.

Ex. 1. By way of illustration, let us take seven elements, 1, 2, 3, 4, 5, 6, 7, connected as indicated in the subjoined synopsis, in which the variations belonging to every element are written on the same line as that element.

Elements.	Variations.				
1	2	3	6	7	
2	1	3	5	6	
3	1	2	4	5	
4	3	5	6	7	
5	2	3	4	6	
6	1	2	4	5	7
7	1	4	6		

The permanences belonging to each element are at once inferred from its variations, e. g. since the variations assigned to element 1 are with the elements 2, 3, 6, 7, it must have permanences with all the remaining elements, viz. 4, 5 (Art. 1), and so on.

From the adopted set of relations, the following cycles of variations are inferred:

Cycles of Variations.

1 2 3; 1 2 6; 1 6 7; 2 3 5; 2 5 6; 3 4 5; 4 5 6; 4 6 7;

all of three elements, and as, under the rule of annotation, no two elements giving a variation can enter the same column, it is clear that every scheme must have three columns at least.

In order to obtain all the schemes, we can begin by writing all the elements in a row,

1, 2, 3, 4, 5, 6, 7,

which gives the scheme having the maximum number of columns; then, by referring to the synopsis of variations, we can reduce the number of columns successively.

Or, we can start with any cycle of variation, e. g.

1, 2, 3

and introduce the remaining elements one by one.

The application of either process will lead to the following general Table of twenty-seven schemes:

1. $\begin{array}{cccc}(1&2&3&5)\\(4&7&6&\ )\end{array}$	10. $\begin{array}{ccccc}(1&2&3&5&6)\\(4&&&7&\ )\end{array}$	19. $\begin{array}{ccccc}(1&2&3&4&5)\\(\ &&6&&7)\end{array}$
2. $\begin{array}{cccc}(1&2&3&5)\\(4&&6&7)\end{array}$	11. $\begin{array}{ccccc}(1&2&3&6&7)\\(5&4&&&\ )\end{array}$	20. $\begin{array}{cccccc}(1&2&3&5&6&7)\\(4&&&&&\ )\end{array}$
3. $\begin{array}{cccc}(1&2&3&7)\\(5&4&6&\ )\end{array}$	12. $\begin{array}{ccccc}(1&2&3&4&6)\\(5&7&&&\ )\end{array}$	21. $\begin{array}{cccccc}(1&2&3&4&6&7)\\(5&&&&&\ )\end{array}$
4. $\begin{array}{cccc}(1&2&3&6)\\(5&4&7&\ )\end{array}$	13. $\begin{array}{ccccc}(1&2&3&4&7)\\(5&&6&&\ )\end{array}$	22. $\begin{array}{cccccc}(1&2&3&5&6&7)\\(\ &4&&&&\ )\end{array}$
5. $\begin{array}{cccc}(1&2&3&4)\\(5&7&6&\ )\end{array}$	14. $\begin{array}{ccccc}(1&2&3&4&6)\\(5&&7&&\ )\end{array}$	23. $\begin{array}{cccccc}(1&2&3&4&5&6)\\(\ &7&&&&\ )\end{array}$
6. $\begin{array}{cccc}(1&2&3&5)\\(\ &4&6&7)\end{array}$	15. $\begin{array}{ccccc}(1&2&3&5&7)\\(\ &4&6&&\ )\end{array}$	24. $\begin{array}{cccccc}(1&2&3&4&5&7)\\(\ &&6&&&\ )\end{array}$
7. $\begin{array}{ccccc}(1&2&3&5&6)\\(4&7&&&\ )\end{array}$	16. $\begin{array}{ccccc}(1&2&3&5&6)\\(\ &4&7&&\ )\end{array}$	25. $\begin{array}{cccccc}(1&2&3&4&5&6)\\(\ &&7&&&\ )\end{array}$
8. $\begin{array}{ccccc}(1&2&3&5&6)\\(4&&7&&\ )\end{array}$	17. $\begin{array}{ccccc}(1&2&3&5&6)\\(\ &4&&7&\ )\end{array}$	26. $\begin{array}{cccccc}(1&2&3&4&5&6)\\(\ &&&&7&\ )\end{array}$
9. $\begin{array}{ccccc}(1&2&3&5&7)\\(4&&6&&\ )\end{array}$	18. $\begin{array}{ccccc}(1&2&3&4&5)\\(\ &7&6&&\ )\end{array}$	27. $(1\ 2\ 3\ 4\ 5\ 6\ 7)$

4. Reverting to the general case, it is evident that the *minimum* number of columns met with in one or several schemes depends essentially upon the *connection* of the elements, that is to say, upon the particular way in which the variations and permanences are distributed. For the purpose of reference to this minimum, it will be convenient to adopt a letter for it. Accordingly, in what follows:

The minimum number of columns will be designated by k.

As an immediate consequence of the meaning of k, in connection with the law of annotation, the maximum cycle of variations cannot involve more than k elements. This consequence may also be stated thus: *Between* $k+1$ *elements taken at random, there exists at least one permanence.*

Observe that if the maximum cycle of variation contains k'-elements, it does not follow that $k=k'$. For, in the above example, the maximum cycle of variation has only three elements, yet $k=4$.

5. If we single out from the general Table all the schemes which have only k or $k+1$ columns, we shall obtain a reduced or partial table which we shall designate by T, and in which every scheme is annotated with the *minimum* number of columns, or the *minimum plus one.*

In the above example, Table T is made up of the schemes 1, 2, ... 19.

Abridged Mode of Writing.

In the sequel we shall represent an aggregate of schemes of an equal number of columns, say μ, by the abridged notation

$$|E_1\ E_2\ E_3 \ldots\ldots E_{\mu-1}\ E_\mu|,$$

where each E stands for a column or, what is the same thing, for some cycle of permanences in the widest sense of the term, including *pairs* or *single elements.* The symbols E_1, E_2, etc., must be understood to represent in turn the different cycles of permanences met with in the various schemes of μ columns, whose collective representation is condensed into the above abridged form.

According to this convention, Table T will be represented by the two abridged forms:

$$|E_1\quad E_2 \ldots\ldots E_k\ E_{k+1}|,$$
$$|.\ E_1\ E_2 \ldots\ldots E_k|,$$

the second of which gives the schemes of k columns considered as schemes of $k+1$ columns, with one blank column indicated by a dot.

It must be well borne in mind that E_1, E_2, etc., being mere symbols, do not represent the same aggregates of elements in both forms.

6. The representative form

$$|E_1\ E_2 \ldots\ldots E_k\ E_{k+1}|$$

admits of one *arbitrary* column, that is to say, in the aggregate of the schemes of $k+1$ columns, every cycle of permanences, including pairs and single elements, will, in some scheme, make up a whole column. This is easily seen as follows:

Let us designate by the notation $\dot{E}$ a cycle of permanences arbitrarily chosen, which might be a permanence-pair or a single element, and let us select any particular scheme of k columns, say

$$|\cdot\ E_1\ E_2 \ldots\ldots E_{k-1}\ E_k|.$$

As this scheme, like every scheme, contains all the elements, we can pick out of the different columns, the elements of which $\dot{E}$ is composed, and place them together in the blank column, which gives the required scheme:

$$|\dot{E}\ E_1'\ E_2' \ldots\ldots E_{k-1}'\ E_k'|.$$

Exceptionally this scheme might reduce to one of k columns. If, for instance, we had taken $\dot{E}=E_1$, our operation on the particular scheme of k columns originally selected would have been nothing else than the shifting of a whole column bodily, thus leaving the scheme unaltered. Nevertheless, we cannot assert *a priori* that a particular cycle of permanences will make up a whole column in some scheme of k columns, e. g. in the above example, where $k=4$, there is no scheme of four columns in which the element 3 makes up a whole column.

With schemes of $k+2$ columns, two columns would become arbitrary, and so on.

We can now condense the two representative forms of Table T into one, viz.

$$|\dot{E}\ E_1\ E_2 \ldots\ldots E_{k-1}\ E_k| \tag{1}$$

for this typical scheme has generally $k+1$ columns, and by making $\dot{E}=0$ we get the schemes of k columns in their abridged notation.

7.—*First Partition.*

Representing, as in Article 1, by $\bar{E}$, the aggregate of the given elements $e_1, e_2, \ldots\ldots e_n$, we partition $\bar{E}$ into two arbitrary distinct sets of elements which by analogy we shall designate by $\bar{A}$ and $\bar{B}$ respectively. We have then symbolically

$$\bar{E} = \bar{A} + \bar{B}.$$

In scheme (1) the symbol E becomes generally $\frac{A}{B}$; likewise the conventional symbol $\dot{E}$ becomes generally $\left(\frac{\dot{A}}{B}\right)$, i. e. a *chosen cycle of permanences containing elements of both sets.* We may, however, without ambiguity, write

$$\dot{E} = \frac{\dot{A}}{\dot{B}},$$

for it is obvious that once the particular cycle of permanences $\dot{A}$ has been chosen, $\dot{B}$ is limited by the law of annotation to such cycles of permanences out of the set $\bar{B}$ as form a cycle of permanences with $\dot{A}$ and *vice versa.* Otherwise told, the two cycles $\dot{A}$ and $\dot{B}$ are arbitrary, subject to the condition of being what may be called *congruent* with one another.

The general form of scheme (1) under the partition is

$$\left|\begin{array}{llll} \dot{A} & A_1 \,\ldots\ldots & A_{k-1} & A_k \\ \dot{B} & B_1 \,\ldots\ldots & B_{k-1} & B_k \end{array}\right|.$$

We can, in the arbitrary column, make separately or conjointly $\dot{A} = 0$, $\dot{B} = 0$. For $\dot{E} = \dot{A}$, $\dot{E} = \dot{B}$, $\dot{E} = 0$ are legitimate values of $\dot{E}$ in scheme (1).

If we make $\dot{B} = 0$, we obtain the following type of schemes:

$$\left|\begin{array}{llllll} \dot{A} & A_1 & A_2 \,\ldots\ldots & A_{k-1} & A_k \\ . & B_1 & B_2 \,\ldots\ldots & B_{k-1} & B_k \end{array}\right., \tag{2}$$

comprising all the schemes of table T in which the elements of the set $\bar{B}$ occupy k columns at most.

The aggregate of such schemes forms a partial table which we shall call the *Table of the first Partition* and designated by T_1.

As all the schemes of T_1 are also schemes of T, we can say that table T_1 is contained in table T.

8. Table T_1 will also contain schemes of the type:

$$\left|\begin{array}{ccccccc} A_1 & A_2 & A_3 & \cdots\cdots & A_{k-1} & A_k & \cdot \\ \cdot & B_1 & B_2 & \cdots\cdots & B_{k-2} & B_{k-1} & B_k \end{array}\right|, \tag{3}$$

in which both sets are written in k columns at most, two of which are different.

In order to justify this assertion, let us first write the scheme in the more convenient form

$$\begin{array}{r|ccccccc|} \text{Columns} & 1 & 2 & 3 \cdots\cdots & k-1 & k & k+1 \\ \hline & A & A & A \cdots\cdots & A & A & \cdot \\ & \cdot & B & B \cdots\cdots & B & B & B \end{array}$$

in which it must be well understood that A, B are mere indeterminate symbols having a *different meaning in each column*, as representing in each an aggregate of different elements. All the symbols A, taken together, make up the set of elements designated by $\bar{A}$; similarly with B.

Now, if in (2), we make $\dot{A} = 0$, we get the general type of the schemes of k columns illustrating the partition, viz:

$$\left|\begin{array}{ccccc} \cdot & A_1 & A_2 & \cdots\cdots & A_k \\ \cdot & B_1 & B_2 & \cdots\cdots & B_k \end{array}\right|.$$

We write it likewise:

$$\begin{array}{|ccccc|} 1 & 2 & 3 & \cdots\cdots & k+1 \\ \hline \cdot & A & A & \cdots\cdots & A \\ \cdot & B & B & \cdots\cdots & B \end{array}$$

and it can represent any particular scheme of k columns. If, now, we remove from any column, e. g. the second, all the elements which belong to the set $\bar{A}$ to place them in the first column—the blank column—we obtain

$$\begin{array}{|cccccc|} 1 & 2 & 3 & \cdots\cdots & k & k+1 \\ \hline A & \cdot & A & \cdots\cdots & A & A \\ \cdot & B & B & \cdots\cdots & B & B \end{array}$$

and this is a scheme of the type (3), for by permuting the second and the $k+1^{\text{th}}$ columns, which leaves the scheme unaltered (Art. 1), we obtain scheme (3) in the new abridged writing.

The two representative schemes of table T_1 of the first partition are then:

$$\begin{array}{|ccccc|}
\multicolumn{1}{c}{1} & 2 & \ldots & k & \multicolumn{1}{c}{k+1} \\ \hline
\dot{A} & A & \ldots & A & A \\
\cdot & B & \ldots & B & B
\end{array}\,, \tag{2}$$

$$\begin{array}{|ccccc|}
\multicolumn{1}{c}{1} & 2 & \ldots & k & \multicolumn{1}{c}{k+1} \\ \hline
A & A & \ldots & A & \cdot \\
\cdot & B & \ldots & B & B
\end{array}\,. \tag{3}$$

Scheme (2) alone admits of an arbitrary column confined to cycles of permanences chosen among the elements of the set $\bar{A}$.

9.—*Second Partition.*

We can proceed to a second partition by sub-partitioning the set $\bar{A}$ into two sets $\bar{A}_1$, $\bar{B}_1$. We have then symbolically

$$\bar{A} = \bar{A}_1 + \bar{B}_1, \quad \bar{E} = \bar{A}_1 + \bar{B}_1 + \bar{B}.$$

In carrying out this second partition in the two representative schemes of the first partition, the symbol B remains unchanged, while each symbol A changes to $\begin{smallmatrix} A_1 \\ B_1 \end{smallmatrix}$. Therefore scheme (2) becomes

$$\begin{array}{|ccccc|}
\multicolumn{1}{c}{1} & 2 & \ldots & k & \multicolumn{1}{c}{k+1} \\ \hline
\dot{A}_1 & A_1 & \ldots & A_1 & A_1 \\
\dot{B}_1 & B_1 & \ldots & B_1 & B_1 \\
\cdot & B & \ldots & B & B
\end{array}$$

the symbols A_1, B_1, B having as before different meanings in the different columns.

We can again make separately or simultaneously $\dot{A}_1 = 0$, $\dot{B}_1 = 0$ (comp. Art. 7). Making $\dot{B}_1 = 0$, we obtain the type of schemes

$$\begin{array}{|ccccc|}
\multicolumn{1}{c}{1} & 2 & \ldots & k & \multicolumn{1}{c}{k+1} \\ \hline
\dot{A}_1 & A_1 & \ldots & A_1 & A_1 \\
\cdot & B_1 & \ldots & B_1 & B_1 \\
\cdot & B & \ldots & B & B
\end{array} \tag{4}$$

Likewise scheme (3) becomes

$$\begin{array}{|ccccc|} 1 & 2 & \cdots & k & k+1 \\ \hline A_1 & A_1 & \cdots & A_1 & \cdot \\ B_1 & B_1 & \cdots & B_1 & \cdot \\ \cdot & B & \cdots & B & B \end{array} . \tag{5}$$

Now, in the same way as we obtained a reduced table T_1 by singling out of table T,[*] all the schemes in which the elements of the set $\bar{B}$ occupied k columns at most, we can obtain another reduced table by singling out of T_1 after the subpartition of $\bar{A}$ into $\bar{A}_1 + \bar{B}_1$ all the schemes in which the elements of the set B_1 occupy k columns at most.

We shall by analogy call this new reduced table the table of the second partition and designate it by T_2.

All the schemes of the types (4) or (5) belong to T_2. But I say that it will, in addition to these, include schemes of the three following types:

$$\begin{array}{|ccccc|} 1 & 2 & \cdots & k & k+1 \\ \hline A_1 & A_1 & \cdots & A_1 & \cdot \\ \cdot & B_1 & \cdots & B_1 & B_1 \\ \cdot & B & \cdots & B & B \end{array} , \tag{6}$$

$$\begin{array}{|ccccc|} 1 & 2 & \cdots & k & k+1 \\ \hline A_1 & A_1 & \cdots & A_1 & \cdot \\ \cdot & B_1 & \cdots & B_1 & B_1 \\ B & B & \cdots & B & \cdot \end{array} , \tag{7}$$

$$\begin{array}{|ccccc|} 1 & 2 & \cdots & k & k+1 \\ \hline A_1 & A_1 & \cdots & A_1 & A_1 \\ B_1 & B_1 & \cdots & B_1 & \cdot \\ \cdot & B & \cdots & B & B \end{array} \tag{8}$$

---

[*] Let us recall that table T is, by definition, made up of the aggregate of schemes of no more than $k+1$ columns.

The proof is as follows:

If in type (4) we make $\bar{A}_1 = 0$, we have a form illustrating the second partition in any scheme of k columns, viz.

$$\begin{array}{c|ccccc|}
 & 1 & 2 & \dots & k & k+1 \\
\hline
 & . & A_1 & \dots & A_1 & A_1 \\
 & . & B_1 & \dots & B_1 & B_1 \\
 & . & B & \dots & B & B
\end{array}\,.$$

Removing the elements belonging to the set $\bar{A}_1$ from the last column to the blank column, we obtain the identical scheme type (6).

To justify the existence of the scheme (7) in table T_2, we first observe that it can be written

$$\begin{array}{c|ccccc|}
 & 1 & 2 & \dots & k & k+1 \\
\hline
 & A_1 & A_1 & \dots & A_1 & . \\
 & B & B & \dots & B & . \\
 & . & B_1 & \dots & B_1 & B_1
\end{array}\,,$$

for the permutation of the two rows is tantamount to a permutation of elements in each column and leaves the scheme unaltered (Art. 2).

This done, it is only necessary to imagine the first partition to have been

$$\bar{E} = \bar{\mathfrak{A}} + \bar{B}_1$$

and the sub-partition

$$\mathfrak{A} = \bar{A}_1 + \bar{B},$$

which gives the same total partition of elements as before, viz.

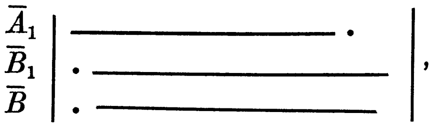
$$\bar{E} = \bar{A}_1 + \bar{B} + \bar{B}_1.$$

Type (7) is then the new form of type (5).

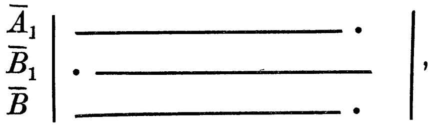
The existence of type (8) follows from the definition of table T_2; for it is obvious that the schemes in which both the set $\bar{B}$ and the set $\bar{B}_1$ take up individually k columns at most, will generally contain the remaining elements, i. e. those of the set $\bar{A}_1$ in all the columns. The same type can also be inferred from the types (5) or (7) by removing into the column devoid of the elements of the set $\bar{A}_1$ one or more elements of that set, which may happen to form a cycle of permanences with the elements of the set $\bar{B}$ or the set $\bar{B}_1$ contained in that column.

The table T_2 of the second partition is then characterized by the five types, (4), (5), (6), (7), (8).

They might also be represented by graphs, e. g.



(6)



(7)

etc.

10. There is no difficulty in generalizing these results. We can form a new reduced table T_3 by singling out of table T_2 the schemes in which the elements of another set $\overline{B}_2$ occupy at most k columns. T_3 will be the table of the third partition of the given elements expressed by the symbolical equation

$$\overline{E} = \overline{A}_2 + \overline{B}_2 + \overline{B}_1 + \overline{B},$$

and this third partition can be conceived to have been obtained from the second partition by sub-partitioning $\overline{A}_1$ into $\overline{A}_2 + \overline{B}_2$.

The typical schemes in any new partition will be inferred from those of the preceding one by breaking up the various symbols into two as may be observed by comparing the first partition with the second. In any representative scheme we may have any number of symbols excluded from one column and the remaining ones from the other, or also one symbol subject to no exclusion, as in type (8) of the second partition, e. g.

$$\begin{array}{|cccc|}
1 & 2 & \ldots\ldots k & k+1 \\
\hline
A_m & A_m & \ldots\ldots A_m & A_m \\
\cdot & B_m & \ldots\ldots B_m & B_m \\
B_{m-1} & B_{m-1} & \ldots\ldots B_{m-1} & \cdot \\
\vdots & \vdots & \ldots\ldots \vdots & \vdots \\
\cdot & B_1 & \ldots\ldots B_1 & B_1 \\
B & B & \ldots\ldots B & \cdot
\end{array}$$

where the symbol A_m can be struck out from the first or from the last column.

If, in any representative scheme, we condense into one symbol all the symbols for which the dot occurs in the same column, we obtain, as should be, a representative scheme belonging to the first partition, for this operation is tantamount to condensing several sets into one, so as to reduce the total number of sets to two.

Remark.—There is only one type of schemes in any partition where one column may be considered arbitrary, viz.

$$\begin{array}{|cccccc|}
1 & 2 & & \ldots & k & k+1 \\ \hline
\dot{A}_m & A_m & & \ldots & A_m & A_m \\
\cdot & B_m & B_m & \ldots & B_m & B_m \\
\cdot & B_{m-1} & B_{m-1} & \ldots & B_{m-1} & B_{m-1} \\
 & \vdots & \vdots & \ldots & \vdots & \vdots \\
\cdot & B_1 & B_1 & \ldots & B_1 & B_1 \\
\cdot & B & B & \ldots & B & B
\end{array}\;.$$

This circumstance will play no part in the sequel, and while it seemed difficult to omit every mention of it, on account of its obviousness, yet no particular stress need be laid upon it.

11. The results arrived at so far can be summed up in the following proposition:

Proposition I.—If elements in any number be arbitrarily connected each to each by one or the other of two reciprocal relations, and if k be the minimum of columns in which they can be annotated (subject to the law given, Art. 2), then if the aggregate of these elements be partitioned into any number of sets arbitrarily composed, it will always be possible to annotate schemes of no more than $k+1$ columns in which the elements belonging to any number of these sets shall be excluded from one column, and the elements belonging to the remainder of the sets shall be excluded from another column.

Otherwise,

In building up schemes of no more than $k+1$ columns by introducing the elements one by one in any order, it will always be possible to obtain schemes in which every element as, and when introduced, shall be excluded from one or from the other of two columns of previously determined ranks.

Remark.—The possibility of such an exclusion is strictly confined to two columns. In the second partition, for instance, we have not been led to any scheme of the type

$$\begin{array}{|ccccccc|}
1 & 2 & 3 & & \ldots\ldots k & k+1 & \\
\hline
\cdot & A_1 & A_1 & & \ldots\ldots A_1 & A_1 & \\
B_1 & \cdot & B_1 & & \ldots\ldots B_1 & B_1 & \\
B & B & \cdot & B & \ldots\ldots B & B &
\end{array}$$

in which the three sets appear, simultaneously excluded from three different columns. Nor is the existence of such a scheme generally possible so long as the connection between the elements remains arbitrary.

Calling *exclusion feature* the general possibility of excluding sets of elements from one or more columns, Proposition I asserts that—

When the connection of the elements remains arbitrary, the exclusion-feature is, generally speaking, restricted to two columns with respect to every element in the aggregate of the schemes of no more than $k+1$ columns, where k is the minimum of columns which any scheme can have.

12. *Ex.* 2.—Let us take thirteen elements connected as follows:

Elements.	Variations.							
1	2	3	4	6	7	8	13	
2	1	3	5	6	8	9		
3	1	2	4	5	7	9		
4	1	3	5	6	8	11		
5	2	3	4	6	7	8	13	
6	1	2	4	5	7	10	11	
7	1	3	5	6	8	9	12	13
8	1	2	4	5	7	9	10	
9	2	3	7	8	10	12		
10	6	8	9	11	13			
11	4	6	10	12				
12	7	9	11					
13	1	5	7	10				

In order to determine k (minimum of columns, Art. 4), we can start with the cycle of variations

$$\begin{array}{ccc} 1 & 2 & 3 \end{array}$$

and introduce the elements one by one in their natural numerical order, putting them down in the smallest number of columns consistent with the synopsis of variations. It is only necessary to take into consideration the variation of each successive element with the preceding elements already annotated.

e. g. From the synopsis, element 4 has variations with elements 1, 3 already annotated; we can then write:

$$\begin{array}{ccc} 1 & 2 & 3 \\ & 4 & \end{array}$$

Element 5 has variations with 2, 3, 4, and we can write

$$\begin{array}{ccc} 1 & 2 & 3 \\ 5 & 4 & \end{array}$$

Proceeding in this way, we shall obtain a *unique* scheme of three columns, viz.

$$\left|\begin{array}{ccc} 1 & 2 & 3 \\ 5 & 4 & 6 \\ 9 & 7 & 8 \\ 11 & 10 & 12 \\ & & 13 \end{array}\right|.$$

We have then $k = 3$.

In order to illustrate the typical schemes of the second partition, let us throw the thirteen elements into three sets:

$$\begin{aligned} \overline{A}_1 &= 1 \quad 2 \quad 3 \quad 11 \quad 13, \\ \overline{B}_1 &= 5 \quad 7 \quad 8 \quad 10, \\ \overline{B} &= 4 \quad 6 \quad 9 \quad 12, \end{aligned}$$

which we can conceive to be derived from a first partition

$$\begin{aligned} \overline{A} &= 1 \quad 2 \quad 3 \quad 5 \quad 7 \quad 8 \quad 10 \quad 11 \quad 13, \\ \overline{B} &= 4 \quad 6 \quad 9 \quad 12, \end{aligned}$$

followed by the sub-partition of $\overline{A}$ into $\overline{A}_1 + \overline{B}_1$, where $\overline{A}_1$ and $\overline{B}_1$ are the sets above given.

If in the scheme of three columns we remove from the first column the elements 5, 9 which belong to $\bar{B}_1$ and $\bar{B}$ respectively, we obtain the scheme

$$\begin{array}{c|cccc|}
\bar{A}_1 & 1 & 2 & 3 & \cdot \\
 & 11 & & 13 & \\[1em]
\bar{B}_1 & \cdot & 7 & 8 & 5 \\
 & & 10 & & \\[1em]
\bar{B} & \cdot & 4 & 12 & 9 \\
 & & & 6 &
\end{array}$$

illustrating the typical scheme (6) of the second partition.

If we only remove element 5 into the fourth column, we get

$$\begin{array}{c|cccc|}
\bar{A}_1 & 1 & 2 & 3 & \cdot \\
 & 11 & & 13 & \\[1em]
\bar{B}_1 & \cdot & 7 & 8 & 5 \\
 & & 10 & & \\[1em]
\bar{B} & 9 & 4 & 6 & \cdot \\
 & & & 12 &
\end{array}$$

illustrating type (7).

If in this last scheme we remove 11 to the fourth column, we obtain a scheme of the type (8), viz.

$$\begin{array}{c|cccc|}
\bar{A}_1 & 1 & 2 & 3 & 11 \\
 & & & 13 & \\[1em]
\bar{B}_1 & \cdot & 7 & 8 & 5 \\
 & & 10 & & \\[1em]
\bar{B} & 9 & 4 & 6 & \cdot \\
 & & & 12 &
\end{array}$$

Remark.—Although always leading to a specimen of types (5)(6)(7), this simple process may not be sufficient to obtain all the schemes of a given type;

e. g. the scheme

$$\begin{array}{c|cccc|}
\bar{A}_1 & 1 & 2 & 3 & \cdot \\
 & & 11 & 13 & \\
\\
\bar{B}_1 & 5 & 7 & 8 & \cdot \\
 & 10 & & & \\
\\
\bar{B} & \cdot & & 6 & 4 \\
 & & & 12 & 9
\end{array}$$

which illustrates type (5), cannot be evolved out of the scheme of three columns by the mere removal of a few elements from one and the same column, but it is easily obtained by introducing the elements successively.

The restriction of the *exclusion-feature* asserted in Proposition I and pointed out in Art. 11, Remark, can also be readily illustrated.

Let it be required to compose a scheme of no more than four columns in which the above sets, $\bar{A}_1$, $\bar{B}_1$, $\bar{B}$, shall be excluded from the first, the second and the third columns respectively.

We can start with the cycle of variation 1 2 3 and as these three elements are to be excluded from the first column as belonging to the set $\bar{A}_1$, we can write them in the six different ways:

$$|\cdot\ 1\ 2\ 3\,|;\quad |\cdot\ 1\ 3\ 2\,|;\quad |\cdot\ 2\ 1\ 3\,|;$$
$$|\cdot\ 2\ 3\ 1\,|;\quad |\cdot\ 3\ 1\ 2\,|;\quad |\cdot\ 3\ 2\ 1\,|;$$

which, in composing schemes, must be examined one after the other.

We begin with $|\cdot\ 1\ 2\ 3\,|$ and introduce the elements in their natural order.

Element 4 has the variations 1, 3 (Synopsis), and being excluded from the third column as belonging to the set $\bar{B}$, can only be put down in the first column. We have then so far

$$\begin{array}{c|cccc|l}
\bar{A}_1 & \cdot & 1 & 2 & 3 & , \\
\bar{B}_1 & & \cdot & & & , \\
\bar{B} & 4 & & \cdot & & .
\end{array}$$

Element 5 has the variations 2, 3, 4, and being excluded from the second column as belonging to the set $\bar{B}_1$, can be placed nowhere in this scheme, which consequently drops out.

The next initial annotation is

$$| \cdot \quad 1 \quad 3 \quad 2 \,|.$$

Here element 4 has two places, giving the two schemes,

$$\begin{array}{c|cccc|}
\bar{A}_1 & \cdot & 1 & 3 & 2 \\
\bar{B}_1 & & \cdot & & \\
\bar{B} & 4 & & \cdot &
\end{array}\,, \qquad
\begin{array}{c|cccc|}
\bar{A}_1 & \cdot & 1 & 3 & 2 \\
\bar{B}_1 & & \cdot & & \\
\bar{B} & & & \cdot & 4
\end{array}\,.$$

Again the first one does not admit of element 5. In the second, 5 has only one place, and we get

$$\begin{array}{c|cccc|}
\bar{A}_1 & \cdot & 1 & 3 & 2 \\
\bar{B}_1 & 5 & \cdot & & \\
\bar{B} & & & \cdot & 4
\end{array}\,,$$

but 6 having the variations 1, 2, 4, 5, and being excluded from the third column as belonging to the set $\bar{B}$, can be placed nowhere, and the scheme is lost.

We must then turn to the third initial annotation of the elements 1, 2, 3 and by trying them all in succession, it will be found that in several schemes we can successfully annotate eight elements, but no more, e. g. we have the scheme

$$\begin{array}{c|cccc|}
\bar{A}_1 & \cdot & 3 & 1 & 2 \\
\bar{B}_1 & 8 & \cdot & 5 & 7 \\
\bar{B} & & 6 & \cdot & 4
\end{array}$$

into which element 9, having the variations 2, 3, 7, 8, and being excluded from the third column as belonging to the set $\bar{B}$, cannot be annotated.

13. To sum up: Whatever be the distribution of variations and permanences among the given elements, we can, by means of the synopsis recording their connection, build up schemes, element by element, of k and $k+1$ columns (k, minimum number of columns, Art. 4), and if, at any stage of the process, we throw the elements already annotated into two distinct arbitrary sets $\bar{A}$ and $\bar{B}$, we know from the results obtained in this chapter and embodied in Proposi-

tion I, that among the schemes so far obtained there will be one or more of the type

$$\begin{array}{cccccc}
1 & 2 & 3 & \ldots & k & k+1 \\
\hline
A & A & A & \ldots & A & \cdot \\
\cdot & B & B & \ldots & B & B
\end{array},$$

and again, it follows therefrom that in introducing the next element into the schemes of this type, it will be possible, in one or more, to exclude said element, at will, from the column which contains no element of the set $\overline{A}$ or from the column which contains no element of the set $\overline{B}$; for this additional element might have been *a priori* assigned to the set $\overline{A}$ or to the set $\overline{B}$.

CHAPTER II.

14. Hitherto the connection between the elements was purely arbitrary. We shall now examine what restrictions must be introduced in order to obtain a full extension of the exclusion-feature described in Proposition I.

k being as before the minimum number of columns met with in any scheme, (Art. 4), the desired extension consists in the possibility of composing schemes of $k+1$ columns in which $k+1$ arbitrary sets of elements shall be excluded respectively from $k+1$ different columns.

Adhering to former notations, the symbol of such schemes would be (Chapter I, Art. 8),

$$\begin{array}{ccccccc}
1 & 2 & 3 & \ldots & k-1 & k & k+1 \\
\hline
\cdot & A_{k-1} & A_{k-1} & \ldots & A_{k-1} & A_{k-1} & A_{k-1} \\
B_{k-1} & \cdot & B_{k-1} & \ldots & B_{k-1} & B_{k-1} & B_{k-1} \\
B_{k-2} & B_{k-2} & \cdot & \ldots & B_{k-2} & B_{k-2} & B_{k-2} \\
\vdots & & & & & & \vdots \\
B_2 & B_2 & B_2 & \ldots & \cdot & B_2 & B_2 \\
B_1 & B_1 & B_1 & \ldots & B_1 & \cdot & B_1 \\
B & B & B & \ldots & B & B & \cdot
\end{array}, \qquad \text{(S)}$$

or, more generally,

$$\begin{array}{|ccccccc|}
\multicolumn{1}{c}{1} & \multicolumn{1}{c}{2} & \multicolumn{1}{c}{3} & \multicolumn{1}{c}{\ldots\ldots k-1} & \multicolumn{1}{c}{k} & \multicolumn{1}{c}{k+1} & \multicolumn{1}{c}{} \\ \hline
A_k & A_k & A_k & \ldots\ldots & A_k & A_k & A_k \\
\bullet & B_k & B_k & \ldots\ldots & B_k & B_k & B_k \\
B_{k-1} & \bullet & B_{k-1} & \ldots\ldots & B_{k-1} & B_{k-1} & B_{k-1} \\
\vdots & & & & & & \vdots \\
B_2 & B_2 & B_2 & \ldots\ldots & \bullet & B_2 & B_2 \\
B_1 & B_1 & B_1 & \ldots\ldots & B_1 & \bullet & B_1 \\
B & B & B & \ldots\ldots & B & B & \bullet
\end{array}, \qquad (S')$$

where one of the sets, viz. $\bar{A}_k$ is subject to no restriction as to annotation.

That such schemes are obtainable I will presently show, but the exposition will be facilitated by the introduction of a few new conventional terms.

15.—*Monogeneous and Polygeneous Elements.*

When schemes are built up element by element, each additional element finds a certain number of variations and of permanences among the elements already annotated.

If the new element has only one variation among the elements already annotated, it will be called *monogeneous;* if it has several, *polygeneous,* and in the latter case, *duogeneous*, *trigeneous*, and, generally, *m-geneous*, according to the number of such variations, or also *polygeneous* of the order m.

16. We can now assert the following fact:

If an additional element be $k-1$-geneous at most, it will be possible to exclude it from any column whatsoever in each of the schemes of $k+1$ columns previously annotated.

The proof is immediate.

By hypothesis, the new element is $k-1$-geneous at most. By definition, it finds in each previous scheme at most $k-1$ variations. The elements giving these variations can then take up at most $k-1$ columns in each scheme. Hence, schemes of $k+1$ columns being allowed, the $k-1$-geneous element can be entered into two columns at least in each scheme. In other words, there are in each scheme two columns at least in which the $k-1$-geneous element finds nothing but permanences.

This told, one of two things must happen.

Either in any scheme the particular column from which it is intended to exclude the new element, contains one of its variations, or it does not.

If it does, the new element is by the law of annotation excluded from that column, and the extra prohibitive condition is superfluous.

If it does not, the new element might be put down in such column, but may be also kept off from it and entered elsewhere, as there is at least one other column in which the new element has no variation. Q. E. D.

The same will be true of another $k-1$-geneous element, and, consequently, of any number of $k-1$-geneous consecutive elements.

As an immediate consequence, if the given elements can be assigned such an order as will make each at most $k-1$-geneous with respect to the preceding ones, then in the annotation of the schemes the possibility of exclusion will be unrestricted.

17. There is no difficulty in obtaining such an aggregate of elements. The easiest way is to start with a cycle of variations of k elements

$$e_1 \quad e_2 \quad e_3 \ldots\ldots e_{k-1} \quad e_k,$$

when, so far, the minimum of columns is assuredly k, and next to add any number of elements

$$e_{k+1}, \quad e_{k+2} \ldots\ldots e_{n-1}, \quad e_n$$

at most $k-1$-geneous, which is done by giving e_{k+1} no more than $k-1$ variations among the preceding elements $e_1, e_2 \ldots\ldots e_k$; in the same way giving e_{k+2} no more than $k-1$ variations among the preceding elements, $e_1, e_2, \ldots\ldots e_k, e_{k+1}$, and so on. The value of k as designating the minimum of columns will remain unaltered throughout, and in the schemes of $k+1$ columns the exclusion-feature, as before understood, will be unrestricted.

18. *Ex.* 3.—As an example, let us resume the thirteen elements of example 2 (Art. 12), in which $k=3$. They are not *duogeneous* ($k-1$-geneous) in their

natural order, but if we strike out the elements 5, 6, 8, 9, 13 the remaining eight elements, 1, 2, 3, 4, 7, 10, 11, 12 are at most *duogeneous* in the order in which they are given, as the synopsis in Ex. 2 shows.

We now add twelve more duogeneous elements, and write down the new synopsis:

SYNOPSIS.

ELEMENTS.	VARIATIONS.	
Duogeneous at most.	With preceding elements.	With subsequent elements.
1	none	2 3 4 7
2	1	3
3	1 2	4 7 14
4	1 3	11
7	1 3	12
10	none	11 16 20
11	4 10	12 15 16
12	7 11	14 18
14	3 12	15 19 22
15	11 14	17 20
16	10 11	17 18 21
17	15 16	19
18	12 16	21
19	17 14	23
20	10 15	22
21	18 16	24
22	14 20	23 25
23	19 22	24
24	21 23	25
25	22 24	none.

We still have $k = 3$, for the synopsis shows that the following scheme of three columns is relevant under the law of annotation:

$$\left|\begin{array}{ccc} 1 & 2 & 3 \\ 10 & 4 & 11 \\ 12 & 7 & 17 \\ 15 & 14 & 18 \\ 19 & 16 & 20 \\ 21 & 23 & 24 \\ 22 & 25 & \end{array}\right| .$$

Take now

$$\begin{array}{lccccc} \bar{A}_2 = & 1 & 2 & 3 & 11 & 25 \\ \bar{B}_2 = & 7 & 10 & 16 & 19 & 23 \\ \bar{B}_1 = & 14 & 15 & 18 & 21 & 24 \\ \bar{B} = & 4 & 12 & 17 & 20 & 22 \end{array}$$

and let it be proposed to work out a scheme of the type:

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline \cdot & A_2 & A_2 & A_2 \\ B_2 & \cdot & B_2 & B_2 \\ B_1 & B_1 & \cdot & B_1 \\ B & B & B & \cdot \end{array} .$$

Starting with the embryo-scheme $| \cdot \quad 1 \quad 2 \quad 3 |$, and introducing the elements one by one in the numerical order of magnitude, we obtain, among others, the scheme:

$$\bar{A}_2 \left|\begin{array}{cccc} \cdot & 1 & 2 & 3 \\ & 25 & & 11 \end{array}\right| ,$$

$$\bar{B}_2 \left|\begin{array}{cccc} 16 & & 7 & 19 \\ 23 & \cdot & 10 & \end{array}\right| ,$$

$$\bar{B}_1 \left|\begin{array}{cccc} 14 & 15 & \cdot & 18 \\ & 21 & & 24 \end{array}\right| ,$$

$$\bar{B} \left|\begin{array}{cccc} 4 & 12 & 17 & \cdot \\ 20 & & 22 & \end{array}\right| .$$

19. The unrestricted exclusion-feature which, as proven, pertains to an aggregate of elements, at most $k-1$-geneous *inter se* is expressed in signs by the two symbolical schemes (S), (S′) of Art. 14. We can in (S′) put $A_k = 0$ in any column, and by condensing into one the two symbols which will then be missing from the same column we fall back on the type (S). In both the symbols can be further sub-divided into any number of symbols.

We can now sum up the results obtained in this chapter in the following proposition:

Proposition II.—If the given elements can be assigned such an order as will make them at most $k-1$-geneous, *inter se*, the exclusion-feature in the aggregate of schemes of $k+1$ columns is unrestricted.

CHAPTER III.

20.—I propose in this chapter to draw some inferences from Proposition I (Ch. 1, Art. 11).

Let $\bar{E}$, as before, be an aggregate of elements arbitrarily connected. Making first partition, we have

$$\bar{E} = \bar{A} + \bar{B},$$

and, as proven in Chapter I, k being the minimum of columns, we shall find among the schemes of no more than $k+1$ columns schemes of the type

$$\begin{array}{|cccc|} \multicolumn{1}{c}{1} & \multicolumn{1}{c}{2 \ldots\ldots} & \multicolumn{1}{c}{k} & \multicolumn{1}{c}{k+1} \\ \hline A & A \ldots\ldots & A & A \\ . & B \ldots\ldots & B & B \end{array}, \tag{2}$$

which is symbolical scheme (2) of the first partition (Ch. I, Art. 7).

N. B.—We have left out the dot over the first symbol A as immaterial, and will do so in the future. (See Remark, Art. 10.)

Let us now introduce a new element α, giving it permanences with all the elements of the set $\bar{A}$ and variations with all the elements of the set $\bar{B}$. We can enter α into the column which contains no element of the set $\bar{B}$ in all the schemes of the type (2) whose aggregate forms what in Ch. I has been called the table of the first partition. We could enter α into no other scheme without augmenting the number of columns, and as it is understood all along that we reject schemes with more than $k+1$ columns, all the schemes not of the type (2) drop out, and after the introduction of α, the general representative scheme of the table under consideration is

$$\begin{array}{|ccccc|}
1 & 2 & \ldots\ldots & k & k+1 \\ \hline
\alpha & & & & \\
A & A & \ldots\ldots & A & A \\
. & B & \ldots\ldots & B & B
\end{array}\ . \tag{2}$$

21.—*First Sub-partition.*

The aggregate of elements being now $\bar{A}+\alpha+\bar{B}$, we introduce a second new element, α_1, giving it any number of variations but with the express condition that they be all taken out of $\bar{A}+\alpha$. The elements thus selected compose a set which we shall call $\bar{B}_1$, the remaining elements, another set $\bar{A}_1$, and we have then

$$\bar{A}+\alpha=\bar{A}_1+\bar{B}_1.$$

By construction, α_1 has nothing but permanences in both the sets, $\bar{A}_1$ and $\bar{B}$ As for α, it may be assigned either to $\bar{A}_1$ or to $\bar{B}_1$; we leave it undecided for the present.

I say now that, *without augmenting the number of columns*, it will be possible to annotate α_1 in some of the schemes in which α has already been annotated.

This, as will be seen, results at once from Proposition I; among others we shall obtain at least one scheme of the form

$$\begin{array}{|ccccc|}
1 & 2 & \ldots\ldots & k & k+1 \\ \hline
 & & & & \alpha_1 \\
A_1 & A_1 & \ldots\ldots & A_1 & A_1 \\
\alpha & & & & \\
B_1 & B_1 & \ldots\ldots & B_1 & . \\
. & B & \ldots\ldots & B & B
\end{array} \tag{8}$$

for, if we erase both α and α_1, we get the scheme type (8) of the second partition, the concrete existence of which in one or more specimens has been proven. (Ch. I, Art. 9.) But in every such scheme α and α_1 are here relevantly annotated, under the rule laid down, viz. all the variations of α make up the set $\overline{B}$, those of α_1, the set B_1.

The elements of the set B do not participate in this second partition, which can be looked upon as a sub-partition of the set $\overline{A}$, increased by α, into $\overline{A}_1$ and $\overline{B}_1$. We have already used the term sub-partition in Ch. I, and it will be convenient to continue to do so, the second partition being the first sub-partition, and so on.

In scheme type (2) of the original partition, we have written α over the symbol A, and in scheme type (8) of the first partition, α_1 over the symbol A_1. This, in order to avoid confusion, and to indicate that α in scheme (2) and α_1 in scheme (8), are *free* elements, i. e. yet unassigned to any particular set.

From the place which α occupies in scheme type (8), it can be conceived to belong either to $\overline{A}_1$ or to $\overline{B}_1$. In the latter case only there is a variation between α and α_1.

If α has been assigned to the set $\overline{A}_1$, then α and α_1 give a permanence, and we shall have such schemes as

$$\begin{array}{|ccccc|}
1 & 2 & \dots & k & k+1 \\ \hline
\alpha_1 & & & & \\
A_1 & A_1 & \dots & A_1 & A_1 \\
\alpha & & & & \\
\cdot & B_1 & \dots & B_1 & B_1 \\
\cdot & B & \dots & B & B
\end{array},$$

which, by erasing α and α_1, reduce to the schemes type (4) of the second partition (Ch. I, Art. 9), the concrete existence of which has been proven. Moreover, α and α_1 are relevantly annotated in respect of their connection both with one another and with the original elements.

22.—*Second Sub-partition.*

The aggregate of elements is now

$$\overline{A}_1 + \alpha_1 + \overline{B}_1 + \overline{B},$$

α being included in $\bar{A}_1$ or in $\bar{B}_1$. We introduce a third new element, α_2, giving it any number of variations, but all taken out of the set $\bar{A}_1 + \alpha_1$.

These variations form a set $\bar{B}_2$, the remainder a set $\bar{A}_2$. We have then

$$\bar{A}_1 + \alpha_1 = \bar{A}_2 + \bar{B}_2,$$

and, by construction, α_2 has permanences with all the elements of the three sets, $\bar{A}_2$, $\bar{B}_1$, $\bar{B}$; α and α_1 may conjointly or separately belong either to $\bar{A}_2$ or to $\bar{B}_2$.

Here again it will be possible, without augmenting the number of columns, to put down α_2 in some of the schemes which already contain α and α_1.

To show it clearly, observe that the connection between α, α_1, α_2 depends on the sets to which α and α_1 have respectively been assigned in the first, and in the second, sub-partitions. Due regard being had to the rule governing the introduction of the new elements, α, α_1, α_2, we find in all the six following cases:

1°. α, α_1, α_2 form a cycle of permanences. Then in the first sub-partition, $A + \alpha = \bar{A}_1 + \bar{B}_1$, α must have been assigned to $\bar{A}_1$, and in the second, $A_1 + \alpha = \bar{A}_2 + \bar{B}_2$, both α and α_1 must have been assigned to $\bar{A}_2$.

2°.	α	α_1.	Permanence.
	α_1	α_2.	Id.
	α_2	α.	Variation.

Then in the first sub-partition, α has been assigned to $\bar{A}_1$. In the second, α_1 has been assigned to $\bar{A}_2$, α to $\bar{B}_2$.

3°.	α	α_1.	Permanence;	α assigned to $\bar{A}_1$,	1st sub-partition.
	α_1	α_2.	Variation;	α_1 " " $\bar{B}_2$,	2d sub-partition.
	α_2	α.	Permanence;	α " " $\bar{A}_2$,	" "

4°.	α	α_1.	Variation;	α assigned to $\bar{B}_1$,	1st sub-partition.
	α_1	α_2.	Permanence;	α_1 " " $\bar{A}_2$,	2d sub-partition.
	α_2	α	Id.		

5°.	α	α_1.	Variation;	α assigned to $\bar{B}_1$,	1st sub-partition.
	α_1	α_2.	Id.	α_1 " " $\bar{B}_2$	2d sub-partition.
	α_2	α.	Permanence.		

6°.	α	α_1.	Permanence;	α assigned to $\bar{A}_1$,	1st sub-partition
	α_1	α_2.	Variation;	α_1 " " $\bar{B}_2$,	2d sub-partition.
	α_2	α.	Id.	α " " $\bar{B}_2$,	

Several otherwise possible cases must here be ruled out, e. g. α, α_1, α_2 cannot form a cycle of variations, for a variation between α and α_1 implies that α has been assigned to $\overline{B}_1$, but, by the standing rule, α_2 has nothing but permanences in the sets $\overline{B}$, $\overline{B}_1$.

A relevant annotation of α_2 in the six cases can be illustrated; among others, in the six corresponding schemes below written:

$$\begin{array}{cccc} 1 & 2 & \ldots & k+1 \\ \hline \multicolumn{1}{|c}{\alpha_2} & & & \multicolumn{1}{c|}{} \\ \multicolumn{1}{|c}{A_2} & A_2 & \ldots & \multicolumn{1}{c|}{A_2} \\ \multicolumn{1}{|c}{\alpha_1} & & & \multicolumn{1}{c|}{} \\ \multicolumn{1}{|c}{\alpha} & & & \multicolumn{1}{c|}{} \\ \multicolumn{1}{|c}{\bullet} & B_2 & \ldots & \multicolumn{1}{c|}{B_2} \\ \multicolumn{1}{|c}{\bullet} & B_1 & \ldots & \multicolumn{1}{c|}{B_1} \\ \multicolumn{1}{|c}{\bullet} & B & \ldots & \multicolumn{1}{c|}{B} \end{array} \quad 1^\circ, \qquad \begin{array}{cccc} 1 & 2 & \ldots & k+1 \\ \hline \multicolumn{1}{|c}{} & & & \multicolumn{1}{c|}{\alpha_2} \\ \multicolumn{1}{|c}{A_2} & A_2 & \ldots & \multicolumn{1}{c|}{A_2} \\ \multicolumn{1}{|c}{} & & & \multicolumn{1}{c|}{\alpha_1} \\ \multicolumn{1}{|c}{B_2} & B_2 & \ldots & \multicolumn{1}{c|}{\bullet} \\ \multicolumn{1}{|c}{\alpha} & & & \multicolumn{1}{c|}{} \\ \multicolumn{1}{|c}{B_1} & B_1 & \ldots & \multicolumn{1}{c|}{\bullet} \\ \multicolumn{1}{|c}{\bullet} & B & \ldots & \multicolumn{1}{c|}{B} \end{array} \quad 2^\circ,$$

$$\begin{array}{cccc} 1 & 2 & \ldots & k+1 \\ \hline \multicolumn{1}{|c}{\alpha_2} & & & \multicolumn{1}{c|}{} \\ \multicolumn{1}{|c}{A_2} & A_2 & \ldots & \multicolumn{1}{c|}{A_2} \\ \multicolumn{1}{|c}{\alpha} & & & \multicolumn{1}{c|}{} \\ \multicolumn{1}{|c}{\bullet} & B_2 & \ldots & \multicolumn{1}{c|}{B_2} \\ \multicolumn{1}{|c}{} & & & \multicolumn{1}{c|}{\alpha_1} \\ \multicolumn{1}{|c}{B_1} & B_1 & \ldots & \multicolumn{1}{c|}{\bullet} \\ \multicolumn{1}{|c}{\bullet} & B & \ldots & \multicolumn{1}{c|}{B} \end{array} \quad 3^\circ, \qquad \begin{array}{cccc} 1 & 2 & \ldots & k+1 \\ \hline \multicolumn{1}{|c}{\alpha_2} & & & \multicolumn{1}{c|}{} \\ \multicolumn{1}{|c}{A_2} & A_2 & \ldots & \multicolumn{1}{c|}{A_2} \\ \multicolumn{1}{|c}{} & & & \multicolumn{1}{c|}{\alpha_1} \\ \multicolumn{1}{|c}{\bullet} & B_2 & \ldots & \multicolumn{1}{c|}{B_2} \\ \multicolumn{1}{|c}{B_1} & B_1 & \ldots & \multicolumn{1}{c|}{\bullet} \\ \multicolumn{1}{|c}{\alpha} & & & \multicolumn{1}{c|}{} \\ \multicolumn{1}{|c}{\bullet} & B & \ldots & \multicolumn{1}{c|}{B} \end{array} \quad 4^\circ,$$

$$\begin{array}{cccc} 1 & 2 & \ldots & k+1 \\ \hline \multicolumn{1}{|c}{\alpha_2} & & & \multicolumn{1}{c|}{} \\ \multicolumn{1}{|c}{A_2} & A_2 & \ldots & \multicolumn{1}{c|}{A_2} \\ \multicolumn{1}{|c}{\bullet} & B_2 & \ldots & \multicolumn{1}{c|}{B_2} \\ \multicolumn{1}{|c}{} & & & \multicolumn{1}{c|}{\alpha_1} \\ \multicolumn{1}{|c}{B_1} & B_1 & \ldots & \multicolumn{1}{c|}{\bullet} \\ \multicolumn{1}{|c}{\alpha} & & & \multicolumn{1}{c|}{} \\ \multicolumn{1}{|c}{\bullet} & B & \ldots & \multicolumn{1}{c|}{B} \end{array} \quad 5^\circ, \qquad \begin{array}{ccccc} 1 & 2 & \ldots & & k+1 \\ \hline \multicolumn{1}{|c}{} & & & & \multicolumn{1}{c|}{\alpha_2} \\ \multicolumn{1}{|c}{A_2} & A_2 & \ldots & A_2 & \multicolumn{1}{c|}{A_2} \\ \multicolumn{1}{|c}{B_2} & B_2 & \ldots & B_2 & \multicolumn{1}{c|}{\bullet} \\ \multicolumn{1}{|c}{\alpha} & & & & \multicolumn{1}{c|}{} \\ \multicolumn{1}{|c}{\alpha_1} & & & & \multicolumn{1}{c|}{} \\ \multicolumn{1}{|c}{\bullet} & B_1 & \ldots & B_1 & \multicolumn{1}{c|}{B_1} \\ \multicolumn{1}{|c}{\bullet} & B & \ldots & B & \multicolumn{1}{c|}{B} \end{array} \quad 6^\circ.$$

The concrete existence of these symbolical schemes can be inferred from the schemes of the first sub-partition (Art. 21) by cancelling α_2 and condensing the two symbols A_2, B_2 into one A_1, when we fall back on the types (4) and (8) of the first sub-partition. Or directly leaving α, α_1, α_2 out of notice, the schemes actually exist in virtue of Proposition I. Moreover, α, α_1 and α_2 are relevantly annotated in respect of their connection, both with one another and with the original elements.

Observe that some cases can be illustrated by more than one representative scheme, e. g. in case 1° the six schemes are relevant; in case 2°, the schemes (2) and (6), etc.

23.— *Generalization.*

The possibility verified up to three additional elements must now be extended to any number of them.

We first recall the main features of the question (Comp., Arts. 20, 21, 22).

A first partition

$$\overline{E} = \overline{A} + \overline{B}$$

being made, the first additional element α is introduced. It is given permanences with all the elements of the set $\overline{A}$ and variations with all the elements of the set $\overline{B}$.

The sub-partition of rank i is

$$\overline{A}_{i-1} + \alpha_{i-1} = \overline{A}_i + \overline{B}_i.$$

It is made for the purpose of introducing the next additional element α_i, which is given permanences with all the elements of the set $\overline{A}_i$ and variations with all the elements of the set $\overline{B}_i$.

The elements of the set $\overline{B}_i$ do not participate in any further sub-partition, they can receive no further variations.

α_{i-1} can be assigned, at will, to $\overline{A}_i$ or to $\overline{B}_i$. In the latter case alone, it is connected with α_i by a variation.

From the rule governing the introduction of the new elements, as here restated, the following immediate inferences can be drawn:

Inferences.

i. The set $\overline{A}_i$ can include any element α_h, where $h < i$. In that case α_h must have been assigned successively to $\overline{A}_{h+1}$, $\overline{A}_{h+2}$ .... $\overline{A}_{i-1}$, $\overline{A}_i$; for if in any intermediate sub-partition α_h had been assigned to $\overline{B}_j$, where j is one of the numbers $h+1, h+2$, etc., it would not, by the rule, have participated in any further sub-partition, and could not, therefore, belong to $\overline{A}_i$.

Any other additional element α_j, where $j < i$, can also belong to $\overline{A}_i$, and generally, any number of additional elements or α-elements, as we will designate them, for brevity.

ii. All the α-elements belonging to $\overline{A}_i$ form a cycle of permanences. For if within $\overline{A}_i$, α_h and α_j gave a variation, where $h < j < i$, then α_h would belong to $\overline{B}_j$, and could not belong to $\overline{A}_i$.

This cycle of permanences admits also of α_i, as, by construction, every variation of α_i is included in $\overline{B}_i$.

iii. Any element α_i can have variations with any number of α-elements of ranks inferior to its own. For by ii $\overline{A}_{i-1}$ can include any number of previous α-elements, and these can, in the next sub-partition $\overline{A}_{i-1} + \alpha_{i-1} = \overline{A}_i + \overline{B}_i$, be all assigned to $\overline{B}_i$. They form a cycle of permanences by ii.

iv. Any α-element can be connected by a variation with only *one* α-element of a rank *superior* to its own.

For if α_i and $\alpha_{i+\mu}$ give a variation, α_i must have been assigned successively to $\overline{A}_{i+1}$, $\overline{A}_{i+2}$ .... $\overline{A}_{i+\mu-1}$, and, finally, to $\overline{B}_{i+\mu}$, when it can receive no more variations.

v. As a consequence, the first additional element α can have only one variation among the α-elements.

vi. The elements of the sets $\overline{B}$, $\overline{B}_1$, etc., must, in all generality, be assumed to actually occupy k columns in each scheme, from which it follows that the additional elements α, α_1, etc., can, generally speaking, be annotated in only *two* columns.

Assuming now the possibility of relevantly annotating m additional elements, $\alpha, \alpha_1, \ldots. \alpha_{m-1}$ in two columns, whatever be the sets to which they may have been successively and are finally assigned, we shall establish the same possibility with respect to α_m, i. e. with $m+1$ additional elements.

To fix ideas, we take $m = 6$, but the reasoning will be general.

We have then to show that $7 = m + 1$ elements $\alpha, \alpha_1, \ldots\ldots \alpha_6$ can be annotated in two columns under any distribution among the sets of the successive sub-partitions, e. g. the following:

$\alpha_2 \quad \alpha_4 \quad \alpha_5$ assigned to $\bar{A}_6$. Hence $(\alpha_2 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6)$ form a cycle of permanences from ii.

$$\left.\begin{array}{llllll} \alpha_3 & \text{"} & \text{"} & \bar{B}_4; & \alpha_4 \ \alpha_3. & \text{Variation.} \\ \alpha_4 & \text{"} & \text{"} & \bar{B}_2; & \alpha_2 \ \alpha_1. & \text{Id.} \end{array}\right\} \text{Standing rule.}$$

We leave α *purposely* unassigned for the present.

The conventional partition-equations are:

Original partition, $\bar{E} = \bar{A} + \bar{B}$.

1st sub-partition, $\bar{A} + \alpha = \bar{A}_1 + \bar{B}_1$, α unassigned thus far.
2d " $\bar{A}_1 + \alpha_1 = \bar{A}_2 + \bar{B}_2$, α_1 assigned to $\bar{B}_2$.
3d " $\bar{A}_2 + \alpha_2 = \bar{A}_3 + \bar{B}_3$, α_2 " $\bar{A}_3$.
4th " $\bar{A}_3 + \alpha_3 = \bar{A}_4 + \bar{B}_4, \begin{cases} \alpha_3 & \text{"} \quad \bar{B}_4. \\ \alpha_2 & \text{"} \quad \bar{A}_4. \end{cases}$
5th " $\bar{A}_4 + \alpha_4 = \bar{A}_5 + \bar{B}_5, \left\{\begin{array}{ll} \alpha_2 & \text{"} \\ \alpha_4 & \text{"} \end{array}\right\} \bar{A}_5.$
6th " $\bar{A}_5 + \alpha_5 = \bar{A}_6 + \bar{B}_6 \left\{\begin{array}{ll} \alpha_2 & \\ \alpha_4 & \text{"} \\ \alpha_5 & \end{array}\right\} \bar{A}_6.$

But so long as α is unassigned, we have here virtually a distribution of $6 = m$ additional elements $\alpha_1, \alpha_2, \ldots\ldots \alpha_6$ to an original aggregate $\bar{A} = \bar{E} - \bar{B}$. For cancelling α, the above equation of the first sub-partition becomes

$$\bar{A} = \bar{A}_1 + \bar{B}_1,$$

which represents an original partition of the aggregate $\bar{A}$. Likewise, the above equation of the second sub-partition, viz.

$$\bar{A}_1 + \alpha_1 = \bar{A}_2 + \bar{B}_2$$

becomes that of the first sub-partition to an original aggregate $\overline{A}$, and so on, the only difference being that the additional elements have here suffixes equal to their ranks, which is immaterial.

Hence, by hypothesis, there are one or more schemes corresponding to the above distribution of these $6 = m$ elements $\alpha_1, \alpha_2, \dots \alpha_6$ in which they are relevantly annotated in two columns. Let us take any one of them, e. g.

$$\left|\begin{array}{llll}
\alpha_6 & & & \\
A_6 & A_6 \dots\dots & & A_6 \\
\alpha_5 & & & \\
\alpha_4 & & & \\
\alpha_2 & & & \\
\cdot & B_6 \dots\dots & & B_6 \\
\cdot & B_5 \dots\dots & & B_5 \\
\cdot & B_4 \dots\dots & & B_4 \\
 & & & \alpha_3 \\
B_3 & B_3 \dots\dots & B_3 & \cdot \\
\cdot & B_2 \dots\dots & & B_2 \\
 & & & \alpha_1 \\
B_1 & B_1 \dots\dots & B_1 & \cdot
\end{array}\right| . \qquad \text{(S)}$$

If we prove that we can reinstate the set $\overline{B}$ and the element α under any hypothesis as to its connection, we shall have established the existence of a proper scheme to the original aggregate $\overline{E}$ and $7 = m + 1$ additional elements, with an arbitrary distribution of these elements among the sets of the successive sub-partitions.

The proof is immediate.

By vi, α can have at most one variation among the α-elements, and this will happen if it has been finally assigned to some set $\overline{B}_i$, giving it a variation with α_i. Then, in the last written scheme, α can be written in the column first or last in which α_i is not, and the exclusion-column of the set $\overline{B}_i$ is thereby determined.

If, for instance, we take $i=3$, we have the scheme

$$\begin{array}{ccccc}
1 & 2 & \dots\dots & k & k+1 \\
\hline
\multicolumn{1}{|c}{\alpha_6} & & & & \multicolumn{1}{c|}{} \\
\multicolumn{1}{|c}{A_6} & A_6 & \dots\dots & A_6 & \multicolumn{1}{c|}{A_6} \\
\multicolumn{1}{|c}{\alpha_5} & & & & \multicolumn{1}{c|}{} \\
\multicolumn{1}{|c}{\alpha_4} & & & & \multicolumn{1}{c|}{} \\
\multicolumn{1}{|c}{\alpha_2} & & & & \multicolumn{1}{c|}{} \\
\multicolumn{1}{|c}{\cdot} & B_6 & \dots\dots & B_6 & \multicolumn{1}{c|}{B_6} \\
\multicolumn{1}{|c}{\cdot} & B_5 & \dots\dots & B_5 & \multicolumn{1}{c|}{B_5} \\
\multicolumn{1}{|c}{\cdot} & B_4 & \dots\dots & B_4 & \multicolumn{1}{c|}{B_4} \\
\multicolumn{1}{|c}{} & & & & \multicolumn{1}{c|}{\alpha_3} \\
\multicolumn{1}{|c}{B_3} & B_3 & \dots\dots & B_3 & \multicolumn{1}{c|}{\cdot} \\
\multicolumn{1}{|c}{\alpha} & & & & \multicolumn{1}{c|}{} \\
\multicolumn{1}{|c}{\cdot} & B_2 & \dots\dots & B_2 & \multicolumn{1}{c|}{B_2} \\
\multicolumn{1}{|c}{} & & & & \multicolumn{1}{c|}{\alpha_1} \\
\multicolumn{1}{|c}{B_1} & B_1 & \dots\dots & B_1 & \multicolumn{1}{c|}{\cdot} \\
\multicolumn{1}{|c}{\cdot} & B & \dots\dots & B & \multicolumn{1}{c|}{B}
\end{array} \qquad (S_1)$$

If α has been finally assigned to the set $\overline{A}_m$, here $\overline{A}_6$, it has *no variation* among the α-elements and can be entered *ad lib.* into the first or the last column, giving the two schemes:

$$\begin{array}{ccccc}
1 & & \dots\dots & & k+1 \\
\hline
\multicolumn{1}{|c}{\alpha_6} & & & & \multicolumn{1}{c|}{} \\
\multicolumn{1}{|c}{A_6} & A_6 & \dots\dots & A_6 & \multicolumn{1}{c|}{A_6} \\
\multicolumn{1}{|c}{\alpha_5} & & & & \multicolumn{1}{c|}{} \\
\multicolumn{1}{|c}{\alpha_4} & & & & \multicolumn{1}{c|}{} \\
\multicolumn{1}{|c}{\alpha_2} & & & & \multicolumn{1}{c|}{} \\
\multicolumn{1}{|c}{\alpha} & & & & \multicolumn{1}{c|}{} \\
\multicolumn{1}{|c}{\cdot} & B_6 & \dots\dots & B_6 & \multicolumn{1}{c|}{B_6} \\
\multicolumn{1}{|c}{\cdot} & B_5 & \dots\dots & B_5 & \multicolumn{1}{c|}{B_5} \\
\multicolumn{1}{|c}{\cdot} & B_4 & \dots\dots & B_4 & \multicolumn{1}{c|}{\cdot} \\
\multicolumn{1}{|c}{} & & & & \multicolumn{1}{c|}{\alpha_3} \\
\multicolumn{1}{|c}{B_3} & B_3 & \dots\dots & B_3 & \multicolumn{1}{c|}{\cdot} \\
\multicolumn{1}{|c}{\cdot} & B_2 & \dots\dots & B_2 & \multicolumn{1}{c|}{B_2} \\
\multicolumn{1}{|c}{} & & & & \multicolumn{1}{c|}{\alpha_1} \\
\multicolumn{1}{|c}{B_1} & B_1 & \dots\dots & B_1 & \multicolumn{1}{c|}{\cdot} \\
\multicolumn{1}{|c}{\cdot} & B & \dots & B & \multicolumn{1}{c|}{B}
\end{array} \qquad (S_2)$$

$$\begin{array}{c|cccc|}
1 & 2 & \ldots & k & k+1 \\
\hline
\alpha^6 & & & & \\
A_6 & A_6 & \ldots & A_6 & A_6 \\
\alpha_5 & & \ldots & & \alpha \\
\alpha_4 & & & & \\
\alpha_2 & & & & \\
\cdot & B_6 & \ldots & B_6 & B_6 \\
\cdot & B_5 & \ldots & B_5 & B_5 \\
\cdot & B_4 & \ldots & B_4 & B_4 \\
 & & & & \alpha_3 \\
B_3 & B_3 & \ldots & B_3 & \cdot \\
\cdot & B_2 & \ldots & B_2 & B_2 \\
 & & & & \alpha_1 \\
B_1 & B_1 & \ldots & B_1 & \cdot \\
B & B & \ldots & B & \cdot
\end{array} \qquad (S_3)$$

The actual existence of these schemes can be vindicated *a posteriori* as in the second sub-partition (Art. 22). If we cancel the α-elements, the three schemes reduce to two and each has at least one actual specimen by Proposition I. On the other hand, the additional elements are correctly annotated in respect of their connection both with one another and with the original elements.

24.—*Definitions of the Terms "Outer Group," "Inner Group."*

We shall generally call "*outer group*" the aggregate of the elements which, in the process of the successive introduction of elements, remain susceptible of receiving new variations.

We shall call "*inner group*" the aggregate of the elements debarred from that possibility, whether arbitrarily or in consequence of some standing rule or hypothesis.

Here, under the rule laid down for the introduction of additional elements, after each new sub-partition, viz.

$$\overline{A}_{i-1} + \alpha_{i-1} = \overline{A}_i + \overline{B}_i,$$

made for the purpose of adding one or more element α_i, the outer group is

$$\bar{A}_i + \alpha_i,$$

the inner group, $\qquad \bar{B}_i + \bar{B}_{i-1} \ldots . \bar{B}_1 + \bar{B}.$

25. The introduction of these two new terms will facilitate the statement of the following facts:

I.—All the α-elements belonging to the outer group can be written in one and the same column.

This may be inferred from schemes (S) and (S_2) (Art. 23). If, before reinstating α into S, all the α-elements of the outer group could be written in the first column as in (S), α also can, as shown in (S_2).

A direct proof is also easily obtained.

Admit the assertion with m α-elements, $\alpha, \alpha_1, \ldots . \alpha_{m-1}$, and let α_p, α_q, etc., be the α-elements which have been finally assigned to $\bar{A}_{m-1}$; the α-elements of the outer group are then $\alpha_{m-1}, \alpha_p, \alpha_q$, etc., and we have, by hypothesis, the scheme

$$\left|\begin{array}{llll}
\alpha_{m-1} & & & \\
A_{m-1} & A_{m-1} & \ldots . & A_{m-1} \\
\alpha_p & & & \\
\alpha_q & & & \\
\cdot & B_{m-1} & \ldots . & B_{m-1} \\
\cdot & B_p & \ldots . & B_p \\
\cdot & B_q & \ldots . & B_q \\
\text{etc.} & & &
\end{array}\right| .$$

In the next sub-partition

$$\bar{A}_{m-1} + \alpha_{m-1} = \bar{A}_m + \bar{B}_m,$$

we can assign α_{m-1} either to $\bar{A}_m$ and the outer group or to $\bar{B}_m$ and the inner group, keeping in both cases $\alpha_p\,\alpha_q$, etc., in the outer group.

In the first case we have

$$\begin{array}{|llll|}
1 & 2 & \dots & k+1 \\
\hline
\alpha_m & & & \\
A_m & A_m & \dots\dots & A_m \\
\alpha_{m-1} & & & \\
\bullet & B_m & \dots\dots & B_m \\
\alpha_p & & & \\
\alpha_q & & & \\
\text{etc.} & & & \\
\bullet & B_{m-1} & \dots\dots & B_{m-1} \\
\bullet & B_p & \dots\dots & B_p \\
\bullet & B_q & \dots\dots & B_q \\
\text{etc.} & & &
\end{array} .$$

In the second case,

$$\begin{array}{|lllll|}
1 & 2 & \dots\dots & & k+1 \\
\hline
\alpha_m & & & & \\
A_m & A_m & \dots\dots & A_m & A_m \\
\alpha_p & & & & \\
\alpha_q & & & & \\
\text{etc.} & & & & \\
\bullet & B_m & \dots\dots & B_m & B_m \\
 & & & & \alpha_{m-1} \\
B_{m-1} & B_{m-1} & \dots\dots & B_{m-1} & \bullet \\
\bullet & B_p & \dots\dots & B_p & B_p \\
\bullet & B_q & \dots\dots & B_q & B_q \\
\text{etc.} & & & &
\end{array} ,$$

in both cases the assertion is justified with one more α-element.

II.—The elements of the outer group can be written in k columns.

For, in either of the two last written schemes, we can put $A_m = 0$ in the last column by Proposition I independently from the connection of the α-elements.

III.—The elements of the outer group continue to exhibit the exclusion-feature as defined Proposition I.

Before proceeding to the proof, let us observe that, after the introduction of the α-elements, the minimum of columns has became $k+1$; therefore, Proposition I does not hold in the aggregate of schemes under consideration; one more column would now be necessary. Neithor does it hold with respect to the α-elements *inter se*, for with respect to these we have $k=2$, and they are only allowed two columns. But, as seen in II, the elements of the outer group comprising an arbitrary set $\overline{A}_m$, made up of original elements and of any number of additional elements, α_p, α_q, etc., are in some schemes actually written in k-columns. Therefore, Proposition I would apply to the outer group if annotated separately in $k+1$ columns, and the condition involved in III is that we shall always find among the schemes resulting from the addition, under the rule laid down, of any number of elements α, α_1, etc., all the schemes which are necessary to illustrate the restricted exclusion-feature with regard to the elements of the outer group in all possible cases. Below a direct and general proof.

26.—*Demonstration of Assertion* III.

From the first wording of Proposition I (Ch. I, Art. 11), the exclusion-feature will be established with respect to the *elements of the outer group* if, throwing them into two sets, $\overline{G}$, $\overline{G}'$, arbitrarily composed, we prove that in the aggregate of the schemes obtained after the addition of any number m of the new elements, $\alpha, \alpha_1, \ldots. \alpha_{m-1}$, there exist schemes in which the elements composing $\overline{G}$, are written in k columns only, and the elements composing $\overline{G}'$, also in k columns, one of which is different; otherwise expressed schemes in which the elements of $\overline{G}$ and of $\overline{G}'$ are *missing* from two different columns, which can be taken as the first and the $k+1^{\text{th}}$.

The elements of the outer group $\overline{A}_m + \alpha_m$ are of two kinds: firstly, elements pertaining to the original aggregate $\overline{E}$, and, secondly, additional elements, viz. α_m and α_p, α_q, $\ldots.$ α_r, which have been finally assigned to $\overline{A}_m$.

Similarly, the two arbitrary sets, $\overline{G}$, $\overline{G}'$, will contain original and additional elements. One of the two sets $\overline{G}$ or $\overline{G}'$ must include α_m. Let us write

$$\overline{G} = \overline{C} + \alpha_p + \alpha_q \ldots. \alpha_r + \alpha_m;$$

$$\overline{G}' = \overline{C}' + \alpha_{p'} + \alpha_{q'} + \ldots. \alpha_{r'},$$

$\bar{C}$ and $\bar{C}'$ being exclusively composed of original elements all contained in $\bar{A}_m$. The suffixes $p, q', \ldots . r$; $p', q', \ldots . r'$, are arbitrary except, of course, that every one of them is $< m$. One of them may be $= 0$; the corresponding element α_0 being in our notation identical with α.

To justify assertion III, we must find at least one scheme in which the elements composing $\bar{G}$ will be missing from say, the last column, and those composing $\bar{G}'$ from the first.

If such a scheme exists, the elements of the outer group must appear in it as shown in the following *embryo-scheme:*

$$\begin{array}{|ccccc|}
\multicolumn{1}{c}{1} & 2 & \ldots . & k & \multicolumn{1}{c}{k+1} \\ \hline
\alpha_m & & & & \\
C & C & \ldots . & C & \cdot \\
\cdot & C' & \ldots . & C' & C' \\
\alpha_p & & \ldots . & & \alpha_{p'} \\
\alpha_q & & \ldots . & & \alpha_{q'} \\
\alpha_r & & \ldots . & & \alpha_{r'} \\
\cdot & B_p & \ldots . & & B_p \\
\cdot & B_q & \ldots . & & B_q \\
\cdot & B_r & \ldots . & \cdot & B_r \\
B_{p'} & & \ldots & B_{p'} & \cdot \\
B_{q'} & & \ldots . & B_{q'} & \cdot \\
B_{r'} & & \ldots . & B_{r'} & \cdot
\end{array},$$

and we have only to prove that the elements of the inner group can be relevantly inserted into it.

Let α_h be the additional element of the inner group which has the *highest suffix*. Being in the inner group, it belongs to some B-set (Art. 24), and has a variation with one, and only one, α-element of a rank superior to its own. (Art. 23, Inference iv.) This α-element is then in the outer group. Moreover, α_h has nothing but permanences with—

1°. The α-elements of the outer group whose ranks are inferior to its own.

For if α_h had a variation, e. g. with α_r, where $r < h$, α_r would belong to $\bar{B}_h$, a set of the inner group, contrary to our hypothesis.

2°. The elements involved in the symbols already written, viz. C, C', B_p, B_q, etc.

For, by the standing rule, the variations of α_h make up the set $\overline{B}_h$ and the corresponding symbol, B_h is not yet written in our scheme.

In conclusion, α_h has one, and only one, variation in the scheme as written so far, and can be entered into the column first or last, in which its variation-element is not.

If it be α_p for instance, we can write α_h in the last column with the symbol B_p, and at the same time the symbol B_h in all the columns *except* the last. We have then added one more α-element and one more line of symbols to our embryo-scheme, which now reads:

$$\begin{array}{|ccccc|}
1 & 2 & \ldots\ldots & k & k+1 \\
\hline
\alpha_m & & & & \\
C & C & \ldots\ldots & C & \cdot \\
\cdot & C' & \ldots\ldots & C' & C' \\
\alpha_p & & \ldots\ldots & & \alpha_{p'} \\
\alpha_q & & \ldots\ldots & & \alpha_{q'} \\
\cdot & B_p & \ldots\ldots & & B_p \\
\text{etc.} & & & & \alpha_h \\
B_{p'} & B_{p'} & \ldots\ldots & B_{p'} & \cdot \\
\text{etc.} & & & & \\
B_h & B_h & \ldots\ldots & B_h & \cdot
\end{array}\ .$$

α_h being disposed of, let α_i be the additional element of the inner group whose suffix is now greatest. α_i, like α_h, belongs to some B-set, and has a variation with an α-element whose suffix is $> i$. It may be α_h or an α-element of the outer group. In any event α_i, just as before, can have only one variation among the elements already annotated. Hence, it can be placed in the column first or last in which its variation-element does not occur.

If, to fix ideas, we take α_h to be the variation-element of α_i, we must enter α_i into the first column with the symbol B_h, and exclude the symbol B_i from the first column. This done, we have again added one more α-element and one more line of symbols to our scheme, which now reads:

$$\begin{array}{|ccccc|}
1 & 2 & \ldots\ldots & k & k+1 \\
\hline
\alpha_m & & & & \\
C & C & \ldots\ldots & C & \cdot \\
\cdot & C' & \ldots\ldots & C' & C' \\
\alpha_p & & \ldots\ldots & & \alpha_{p'} \\
\alpha_q & & \ldots\ldots & & \alpha_{q'} \\
\alpha_r & & \ldots\ldots & & \alpha_{r'} \\
\cdot & B_p & \ldots\ldots & B_p & B_p \\
 & & & & \alpha_h \\
\cdot & B_q & \ldots\ldots & B_q & B_q \\
\cdot & B_r & \ldots\ldots & B_r & B_r \\
B_{p'} & B_{p'} & \ldots\ldots & B_{p'} & \cdot \\
B_{q'} & B_{q'} & \ldots\ldots & B_{q'} & \cdot \\
B_{r'} & B_{r'} & \ldots\ldots & B_{r'} & \cdot \\
B_h & B_h & \ldots\ldots & B_h & \cdot \\
\alpha_i & & & & \\
\cdot & B_i & \ldots\ldots & B_i & B_i
\end{array}$$

Proceeding in this way and reinstating the α-elements of the inner group in the order of decreasing ranks, each of them will always find one, and only one, variation among the elements already annotated, and can then be written in the column first or last, which does not contain the α-element, with which it has a variation.

When all the α-elements and the corresponding B-symbols have been annotated, we have a scheme which is *concrete* with respect to the additional elements, as each letter α represents only one element, and *symbolical* with respect to the original elements. The actual existence of such a scheme is justifiable *a posteriori* by the reasoning often made use of before; the symbolical part of the scheme has, from Proposition I, one or more concrete specimens, no matter how the sets $\overline{B}$, $\overline{B}_1$, ... $\overline{B}_{m-1}$, $\overline{C}$, $\overline{C}'$ are composed, and, on the other hand, the additional elements α, α_1, ... α_m are relevantly annotated both *inter se* and in relation to the original elements, as each additional element α_i is written in the column from which the symbol B_i, which represents the aggregate $\overline{B}_i$ of its variations, can be and has been excluded.

Assertion III is thus fully proven.

27. The process of sub-partitioning the outer group can go on as long as there are elements left to operate upon. At some stage of the addition of α-elements, the outer group might happen to be made up of these altogether. In any case at every stage the outer group is composed for elements which in one or more schemes occupy only k columns. Consequently we could introduce an element α_m giving it variations with *all the elements of the outer group.* This is tantamount to making $\bar{A}_m = 0$ in the equation of the m^{th} sub-partition

$$\bar{A}_{m-1} + \alpha_{m-1} = \bar{A}_m + \bar{B}_m .$$

We have then

$$\bar{A}_{m-1} + \alpha_{m-1} = \bar{B}_m ,$$

and the outer group reduces to the single element α_m. The corresponding symbolical scheme has the form

$$\begin{array}{|cccc|}
1 & 2 & \ldots . k & k+1 \\
\hline
\alpha_m & & & \\
\cdot & B_m & \ldots . B_m & B_m \\
 & & & \alpha_{m-1} \\
 & B_{m-1} & \ldots . B_{m-1} & \cdot \\
\text{etc.} & & &
\end{array}$$

28. We can now enunciate the following theorem:

Theorem.—If elements, arbitrarily connected, have been annotated (according to the rule given Chapter I, Art. 2) in k and $k+1$ columns, k being the minimum, it will be possible, without augmenting the number $k+1$ of columns to add any number of elements, each with any number of variations, provided that at each addition, all the variation-elements of the new element pass on to the inner group. (Definition, Art. 24.)

At each addition many schemes may drop out, viz. those in which none of the $k+1$ columns admit of the introduction of the new element, but there will always remain one or more schemes into which it can be entered.

29. In illustrating this theorem, it will be more interesting to show its bearing on maps in connection with what has been called the geographical problem of the four colors.*

---

* A. B. Kempe, B. A. On the geographical problem of the four colors. American Journal of Mathematics, Vol. II.

The problem, as well known, is to determine how many colors are necessary to paint a map so that two adjacent districts are not painted the same color. By adjacent districts are understood those having a common line-boundary, not merely touching in a point, and the map is supposed to be drawn on a singly connected surface, say a sheet of paper.

Now the rule prescribing a change of color takes no account of any attributes, collective or individual, of the districts, whether in number, shape or space-magnitude. As regards the object aimed at, the districts in any map are mere elements connected each to each by one or the other of two reciprocal relations, which may be called:

Variation in the case of adjacent districts,
Permanence in the case of non-adjacent ones.

Here the second relation is simply the direct negative of the first. If we represent the districts by numbers or letters, the notion of coloring them with a certain number of colors under the rule stated, is the same as writing them in a certain number of columns subject to the condition that no two adjacent districts shall have their representative signs in the same column. Each column will then stand for a color.

From this point of view the coloring of a map is only a particular case of the annotation of elements treated of in this chapter and in the preceding ones, particular in that the connection of the elements is no longer arbitrary but subject to restrictions dependent on the configuration of maps, to which, however, such of the results so far obtained as are free from any hypothesis respecting the connection of the elements will be applicable.

The theorem of this chapter for instance, admits of an immediate adaptation.

A map has external districts which are the border districts, giving the map its *contour*. It has also generally internal districts surrounded by others, and completely shut in. To add a new district we can join two points arbitrarily taken on the contour by a line of any shape drawn outside the map and not meeting itself again. Such a line encloses a space which gives a new external district. The two points selected on the contour are the *end points* of the outline of the new district; they may or may not coincide with points which were common to the boundaries of two former districts, and, as a limiting case, may reduce to a single point.

The new external district will generally be *adjacent* to a certain number, say m, of the former border districts. In the nomenclature here adopted, it will have m variations and according as the end-points of its outline have been chosen it will completely cover up:

$m - 2$ districts
$m - 1$ "
m "

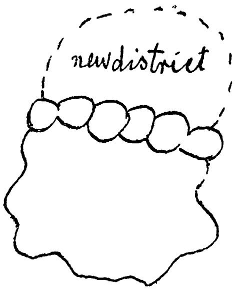


Fig. 1.

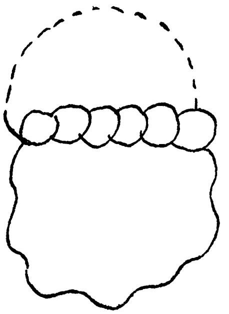


Fig. 2.

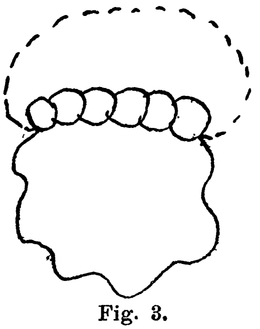


Fig. 3.

It is clear from this graphic process that the variations of every new district are found only among the *external* districts. *Internal* districts can receive no further variations.

Hence, bearing in mind the definitions of Art. 24, the external and internal districts in a map are respectively what we have called the elements of the *outer* and of the *inner* group, which terms have here an obvious concrete meaning.

Every map can obviously be drawn through super-addition of external districts. Hence from the remarks just made if we want the ideal elements hitherto dealt with, to become *symbols* for the districts of a map we must lay down the rule that whenever a polygeneous element (Ch. II, Art. 15) of the order m is added, $m - 2$ at least of its variation-elements pass on to the inner group (Ch. III, Art. 24).

In Fig. 3 the requirements of the theorem are fulfilled as all the variation districts of the new district have passed on to the inner group.

The inference to be drawn therefrom is the following:

Suppose it has been ascertained that the minimum of colors necessary to paint a certain particular map is k. The map can *a fortiori* be painted with $k + 1$ colors. Then without augmenting the number $k + 1$ of colors we can add any number of districts drawn as in Fig. 3, using only two of the colors for them.

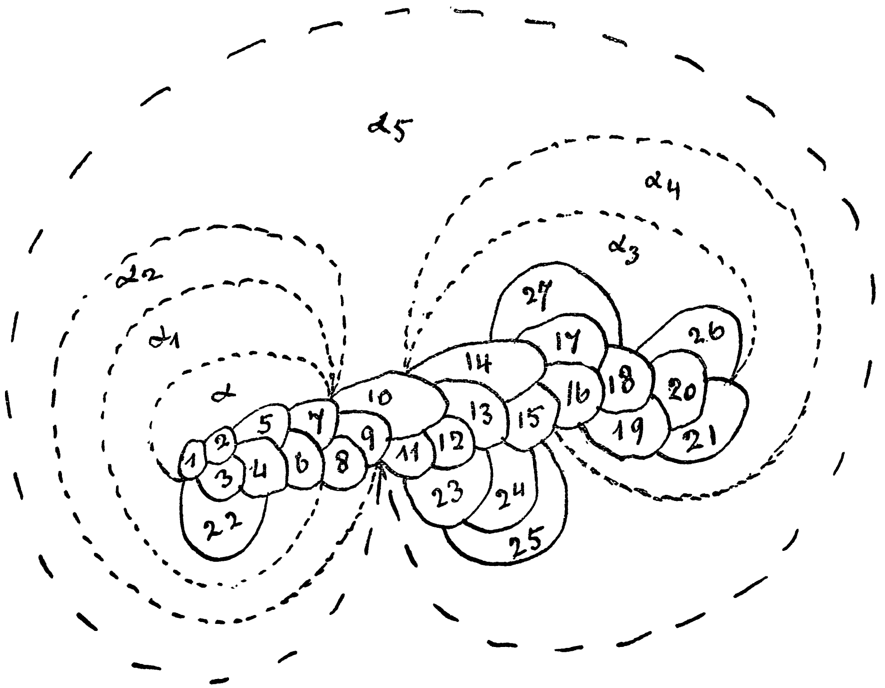


Fig. 4.

30.—*Example* 4.

In the map represented Fig. 4, the districts α, α_1, α_2, α_3, α_4, α_5 fulfil the requirements of the Theorem, Art. 28. Hence if the elements numbered 1, 2 . . . . 26, 27 can be annotated in three columns, it will be possible to annotate the whole map in four columns; i. e. to paint it with four colors.

As regards the partial aggregate of elements, 1, 2, ... 20, 21, we certainly have $k = 3$. For 1, 2, 3, form a cycle of variations. The next element, 4, has, when introduced, only two variations, viz. 2 and 3; the next one, 5, when introduced, only two, viz. 2 and 4, etc. In fact, all the elements, 4, 5, ... 20, 21 are *duogeneous* (Ch. II, Art. 15), and we obtain at once the unique scheme of three columns here given, in which we have repeated the elements, so that each line may show the successive cycles of variations.

I	II	III
1	2	3
4	2	3
4	2	5
4	6	5
7	6	5
7	6	8
7	9	8
7	9	10
11	9	10

,

I	II	III
11	12	10
13	12	10
13	14	10
13	14	15
16	14	15
16	14	17
16	18	17
16	18	19
20	18	19
20	21	19

.

The remaining elements

22, 23, 24, 25, 26, 27

are all trigeneous, and follow the type of Fig. 1. Element 22 has, when added, the variations 1, 3, 4, shutting in element 3, which drops into the inner group; 23 has the variations 11, 12, 13, element 12 passing on to the inner group, etc. But the three variations of every one of these remaining elements always occupy two columns only of the scheme so far written, which circumstance enables us to enter them successively into the same scheme of three columns, as here done, each bracket showing the three variations of the elements as and when added:

I	II	III.
1 4	22	3
11 13	12	23
13	24	23 15
25	24	23 15
20	18 21	26
27	18 14	17

We have then $k = 3$ for the elements numbered 1, 2, ... 26, 27; hence, the whole map can be annotated in four columns; in other words, be painted with four colors, *two* only being required for the α-districts.

31. Below the detail of the operations (comp. Arts. 23, 24).

Original Aggregate of Elements $\bar{E} = 1, 2, \ldots . 26, 27.$

We have here from the start an outer and an inner group, viz.

Outer group: $\bar{E}' = 1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 14, 15, 16, 18, 19, 21, 22, 23, 25, 26, 27.$

Inner group: $\bar{E}'' = 3, 12, 13, 17, 20, 24.$

A first partition must now be made in order to introduce the first additional element α; but as its variations can only be contained in the original outer group $\bar{E}'$, we have here

First partition: $\bar{E}' = \bar{A} + \bar{B},$

$\bar{A} = 4, 6, 8, 9, 10, 11, 14, 15, 16, 18, 19, 21, 22, 23, 25, 26, 27;$

$\bar{B} = 1, 2, 5, 7$ variation-elements of α.

Writing α in the scheme of three columns, we have symbolically

$$\left|\begin{array}{cccc}
\alpha & & & \\
\cdot & A & A & A \\
\cdot & B & B & B \\
\cdot & E' & E'' & E''
\end{array}\right| .$$

Observe that the elements of the set $\bar{E}''$ might be written in four columns. The above scheme is explicitly

$$\left.\begin{array}{c|cccc||}
 & \alpha & & & \\
 & & 4 & 6 & \\
 & & 11 & 9 & 8 \\
 & & & & 10 \\
\bar{A} & & 16 & 14 & 15 \\
 & & & 18 & \\
 & & 25 & 21 & 19 \\
 & & 27 & 22 & 23 \\
 & & & & 26
\end{array}\right\} \text{outer group, } \bar{A}+\alpha$$

$$\left.\begin{array}{c|cccc||}
\bar{B} & \cdot & 1 & 2 & 5 \\
 & \cdot & 7 & & \\
\hline
\bar{E}'' & & 13 & 12 & 3 \\
 & & 20 & 24 & 17
\end{array}\right\} \text{inner group, } \bar{B}+\bar{E}''$$

First sub-partition:

$\bar{A}+\alpha=\bar{A}_1=\bar{B}_1$,

$\bar{A}_1=8,\ 9,\ 10,\ 11,\ 14,\ 15,\ 16\ 18,\ 19,\ 21,\ 23,\ 25,\ 26,\ 27$;

$\bar{B}_1=\alpha,\ 4,\ 6,\ 22$, variation-elements of α_1.

Representative scheme:

$$\left|\begin{array}{cccc}
 & & & \alpha_1 \\
A_1 & A_1 & A_1 & A_1 \\
B_1 & B_1 & B_1 & \cdot \\
\alpha & & & \\
\cdot & B & B & B \\
E' & E'' & E' & E''
\end{array}\right| .$$

Specimen:

$$\left.\begin{array}{c|cccc}
 & & & & \alpha_1 \\
 & & 11 & 9 & 8 \\
 & & 16 & 14 & 10 \\
\overline{A}_1 & & 25 & 18 & 15 \\
 & & 27 & 21 & 19 \\
 & & & & 23 \\
 & & & & 26
\end{array}\right\}\text{ outer group, } \overline{A}_1+\alpha_1;$$

$$\left.\begin{array}{c|cccc}
\overline{B}_1 & \alpha & 4 & 6 & \cdot \\
 & & & 22 & \cdot \\
\hline
\overline{B} & \cdot & 1 & 2 & 5 \\
 & \cdot & 7 & & \\
\hline
\overline{E}'' & & 13 & 12 & 3 \\
 & & 20 & 24 & 17
\end{array}\right\}\text{ inner group, } \overline{B}_1+\overline{B}+\overline{E}''.$$

Second sub-partition:

$$\overline{A}_1+\alpha_1=\overline{A}_2+\overline{B}_2,$$

$$\overline{A}_2=10,\ 11,\ 14,\ 15,\ 16,\ 18,\ 19,\ 21,\ 23,\ 25,\ 26,\ 27;$$

$$\overline{B}_2=\alpha_1\ 8,\ 9,\ \text{variation-elements of } \alpha_2.$$

Representative scheme:

$$\left|\begin{array}{cccc}
\alpha_2 & & & \\
A_2 & A_2 & A_2 & A_2 \\
\cdot & B_2 & B_2 & B_2 \\
 & & & \alpha_1 \\
B_1 & B_1 & B_1 & \cdot \\
\alpha & & & \\
\cdot & B & B & B \\
E'' & E'' & E'' & E''
\end{array}\right|.$$

Specimen:

$\bar{A}_2$	α_2	11	14	10	outer group, $\bar{A}_2 + \alpha_2$;
		16	18	15	
		25	21	19	
		27		23	
				26	
$\bar{B}_2$	•		9	α_1	inner group, $\bar{B}_2 + \bar{B}_1 + \bar{B} + \bar{E}''$.
	•			8	
$\bar{B}_1$	α	4	6	•	
			22	•	
$\bar{B}$	•	1	2	5	
	•	7			
$\bar{E}''$		13	12	3	
		20	24	17	

Third sub-partition:

$$\bar{A}_2 + \alpha_2 = \bar{A}_3 + \bar{B}_3,$$
$$\bar{A}_3 = \alpha_2\ 10,\ 11,\ 15,\ 16,\ 19,\ 21,\ 23,\ 25\ ;$$
$$\bar{B}_3 = 14,\ 18,\ 26,\ 27,\ \text{variation-elements of } \alpha_3.$$

Representative scheme:

α_3			
A_3	A_3	A_3	A_3
α_2			
•	B_3	B_3	B_3
•	B_2	B_2	B_2
			α_1
B_1	B_1	B_1	•
α			
•	B	B	B
E''	E''	E''	E''

Specimen:

$$\left.\begin{array}{c|cccc} & \alpha_3 & & & \\ & \alpha_2 & 11 & 21 & 10 \\ \overline{A}_3 & & 16 & & 15 \\ & & 25 & & 19 \\ & & & & 23 \end{array}\right\} \text{outer group, } \overline{A}_3 + \alpha_3;$$

$$\left.\begin{array}{c|cccc} \overline{B}_3 & \cdot & 27 & 14 & 26 \\ & \cdot & & 18 & \\ \hline \overline{B}_2 & \cdot & & 9 & \alpha_1 \\ & \cdot & & & 8 \\ \hline \overline{B}_1 & \alpha & 4 & 6 & \cdot \\ & & & 22 & \cdot \\ \hline \overline{B} & \cdot & 1 & 2 & 5 \\ & \cdot & 7 & & \\ \hline \overline{E}'' & & 13 & 12 & 3 \\ & & 20 & 24 & 17 \end{array}\right\} \text{inner group, } \overline{B}_3 + \overline{B}_2 + \overline{B}_1 + \overline{B} + E'$$

Fourth sub-partition:

$$\overline{A}_3 + \alpha_3 = \overline{A}_4 + \overline{B}_4,$$
$$\overline{A}_4 = \alpha_2 \ 10, \ 11, \ 15, \ 23, \ 25;$$
$$\overline{B}_4 = \alpha_3 \ 16, \ 19, \ 21, \ \text{variation-elements of } \alpha_4.$$

Representative scheme:

$$\begin{array}{|cccc|} \hline & & & \alpha_4 \\ A_4 & A_4 & A_4 & A_4 \\ & & & \alpha_2 \\ B_4 & B_4 & B_4 & \cdot \\ \alpha_3 & & & \\ \cdot & B_3 & B_3 & B_3 \\ B_2 & B_2 & B_2 & \cdot \\ \alpha_1 & & & \\ \cdot & B_1 & B_1 & B_1 \\ & & & \alpha \\ B & B & B & \cdot \\ E'' & E'' & E'' & E'' \end{array}$$

Specimen:

$\bar{A}_4$				α_4	outer group, $\bar{A}_4+\alpha_4$;
		11		10	
		25		α_2	
				15	
				23	
$\bar{B}_4$	19	16	21	•	inner group, $\bar{B}_4+\bar{B}_3+\bar{B}_2+\bar{B}_1+\bar{B}+\bar{E}''$.
	α_3				
$\bar{B}_3$	•	27	14		
	•		18	26	
$\bar{B}_2$	8		9	•	
	α_1			•	
$\bar{B}_1$	•	4	6	α	
	•		22		
$\bar{B}$	5	1	2	•	
		7		•	
$\bar{E}''$		13	12	3	
		20	24	17	

Fifth sub-partition:

$$\bar{A}_4+\alpha_4=\bar{A}_5+\bar{B}_5,$$
$$\bar{A}_5=0,$$
$\bar{B}_5=11,\ 23,\ 25,\ 15,\ 10,\ \alpha_2,\ \alpha_4,$ variation-elements of α_5.

Representative scheme:

α			
•	B_5	B_5	B_5
			α_4
			α_2
B_4	B_4	B_4	•
α_3			
•	B_3	B_3	B_3
B_2	B_2	B_2	•
α_1			
•	B_1	B_1	B_1
			α
B	B	B	•
E''	E''	E''	E''

.

Specimen:

$$\begin{array}{c|cccc|l}
 & \alpha_5 & & & & \} \text{ outer group, } \alpha_5; \\
 & \cdot & 11 & & 10 & \\
 & \cdot & 25 & & 15 & \\
\overline{B}_5 & \cdot & & & 23 & \\
 & \cdot & & & \alpha_4 & \\
 & \cdot & & & \alpha_2 & \text{inner group, } \overline{B}_5+\overline{B}_4+\overline{B}_3+\overline{B}_2+\overline{B}_1+\overline{B}+\overline{E}''. \\
\overline{B}_4 & 19 & 16 & 21 & \cdot & \\
 & \alpha & & & \cdot & \\
\end{array}$$

etc., identical with the precedent.

The outer group reducing to the single element, α_5, the process of sub-partitioning is closed, and the whole map stands annotated in four columns, the α-elements in two.

32. If we add two new elements, e_1, e_2, forming a cycle of variations with α_5, the unique external element of our map (Fig. 5), we find ourselves pre-

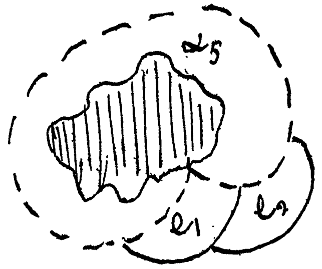


Fig. 5.

cisely in the same situation as in the original map, when we started with the cycle of variations given by the elements 1, 2, 3. We could designate the three elements, e_1, e_2, α_5, in their turn by the numbers 1, 2, 3, and on these elements build up the same map as before, or another, first by adding any quantity of duogeneous elements, thus leaving the minimum of columns $k = 3$ unaltered; next adding, as we did before, polygeneous elements *so chosen as to preserve the same minimum* 3, and, finally, a new set of α-elements fulfilling the requirements of the theorem, i. e. symbolizing districts drawn as in Fig. 3.

In annotating this second map, the elements of the inner group in the first map may be entirely disregarded, and the new map annotated separately without reference to the first. When completed, the two final schemes representing the first and the second map, can be superposed, they having only one element in common, viz. α_5; they then combine into one scheme descriptive of the aggregate of the two maps.

We have supposed, for clearness, all the duogeneous elements to be added first, and next all the polygeneous elements which leave the minimum of columns unaltered, but the two operations can be alternated at will.

If, instead of making $\overline{A}_5 = 0$ in the fifth sub-partition, we had assigned two elements to that set, e. g.

$$\overline{A}_5 = 11,\ 23,$$
$$\overline{B}_5 = \alpha_2,\ \alpha_4,\ 10,\ 15,\ 25,$$

the outer group would have been α_5 11 23 and the final scheme

$$\begin{array}{c|ccc||l}
\overline{A}_5 & \alpha_5 & 11 & 23 & \}\ \text{outer group,} \\
\hline
 & . & 25 & 10 & \\
 & . & & 15 & \\
\overline{B}_5 & . & & \alpha_2 & \}\ \text{inner group.} \\
 & & & \alpha_4 & \\
 & \text{etc.} & & &
\end{array}$$

The elements 11, 23 are connected by a variation, and both with α_5 by a permanence (Fig. 6), but as in the final scheme, these three elements occur in three different columns, we can consider them fictitiously as forming a cycle of variation, and build up a new map on them just as before. When completed, the two final schemes will be superposable as the only three elements they have in common, viz. α_5, 11, 23, can be written in both in the same three columns.

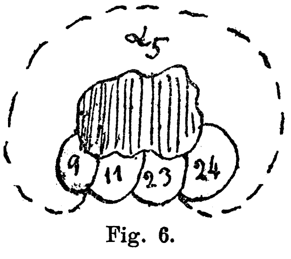


Fig. 6.

A glance at the map (Fig. 6) shows that our fictitious hypothesis is tantamount to tracing the last α-district, viz. α_5 after the type of Fig. 1 instead of Fig. 3, for then the three contour-districts, α_5, 11, 23, will actually form a cycle of variations. In this case, the last α-element does not fulfill the requirements of the theorem, yet no increase in the number of colors is necessary.

Superconstructions of maps in the two cases just recorded can evidently be repeated indefinitely, but if the last α-element left more than two of the original elements on the border, the argument would fail, as on account of the sub-partitions in the second map one cannot assert *a priori* that the elements common to both maps can be made to occupy the same columns in both final schemes.

33. The illustrations here given of the theorem of Art. 28, shows that a great variety of maps can be painted with four colors in consequence of Proposition I alone. That this proposition cannot of itself cover the whole ground is evident, and it is hardly necessary to observe that the example has not been introduced as leading to a solution of the geographical problem, but merely in order to illustrate how general results in the annotation of ideal elements can be applied to maps.

The problem in the annotation of ideal elements to which the general geographical problem corresponds, depends on Proposition II.